6 **Applications of QR**

(1) Solve square linear systems. How?

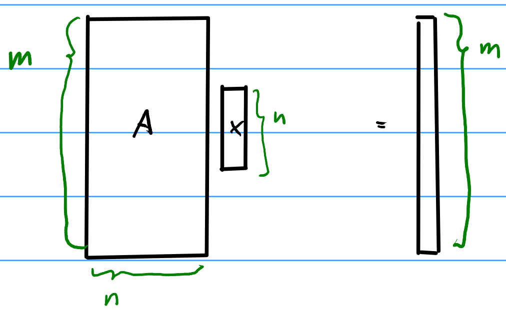
$$Ax = b \rightsquigarrow Q\underbrace{Rx}_{y} = b \qquad O(n^3)$$

$$Qy = b \iff Q^T b = y \qquad O(n^2)$$

$Rx = y$ solvable using backsubstitution $O(n^2)$

This works, but it is practically a little slower than LU.

(2) Solve tall and skinny linear systems. How?



Tall and skinny: $m > n$

More equations than unknowns

A square linear system often only has a single solution.

So applying *more* conditions to the solution will mean...

we have no exact solution. $\longrightarrow$ Get as close as we can.

$\cancel{A\vec{x} = \vec{b}}$ nope

Find $\vec{x}$ so that $\|A\vec{x} - \vec{b}\|_2^2$ is as small as possible.

$\vec{r} = A\vec{x} - \vec{b}$ is called the residual (not zero, but ideally small)

$\|A\vec{x} - \vec{b}\|_2^2 = r_1^2 + \cdots + r_m^2 \longleftarrow$ squares

This is called a (linear) least-squares problem.

Is there other notation for least-squares problems?

All these are equivalent:

- Find $\vec{x}$ so that $\|A\vec{x}-\vec{b}\|_2^2$ is as small as possible.
- $\vec{x} = \arg\min_{\vec{x}} \|A\vec{x}-\vec{b}\|_2^2$
- $Ax \cong b$

And how does QR help with least-squares problems?

$$A = QR$$

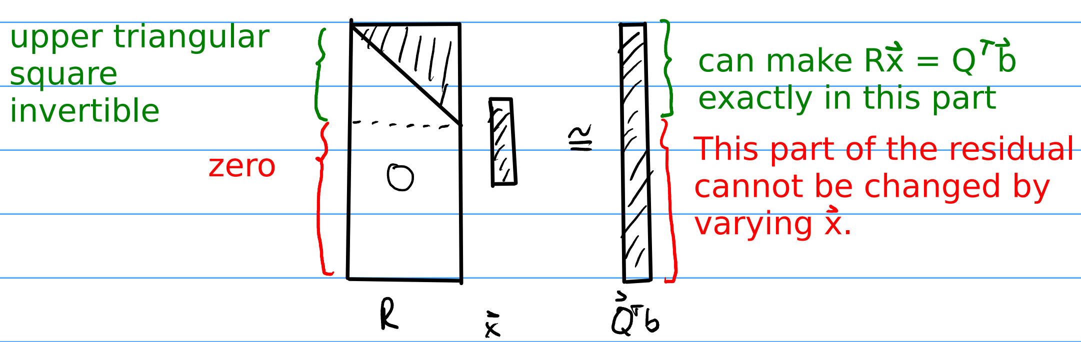
$$\|A\vec{x}-\vec{b}\|_2^2 = \|QR\vec{x}-\vec{b}\|_2^2$$

$$= \|Q^T(QR\vec{x}-\vec{b})\|_2^2$$

$$= \|R\vec{x}-Q^T\vec{b}\|_2^2$$

Since this is *exactly equal* to the expression we wanted to minimize originally, we might as well minimize this.

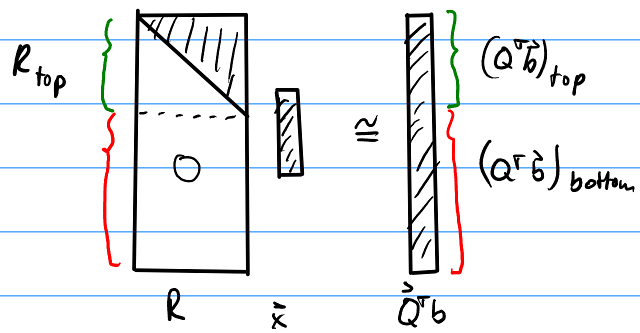
Is this any easier to minimize?



So how do I solve a least-squares problem with QR?

(1) Compute a QR factorization A = QR

(2) Introduce notation:



(3) Solve $R_{top}\vec{x} = (Q^T\vec{b})_{top}$ (using backsubst)

(4) The norm of the residual will be

$$\|Ax-b\|_2 = \|(Q^Tb)_{bottom}\|_2 .$$

What about the "normal equations"?

$A^TAx = A^Tb$ solves the same problem

produces the same x

is not well-behaved numerically

$$\underbrace{\kappa(A^TA)}_{\text{bigger}} \approx \underbrace{\kappa(A)^2}_{\text{big}}$$

bad

Demo: Solving Least-Squares Problems

How can I use least-squares problems for fitting a linear model to data?

Model parameters → (Linear) Model → Predicted Output

Problem: Want to know → Model parameters

Independent Variables → (Linear) Model

<u>Data fitting</u>:

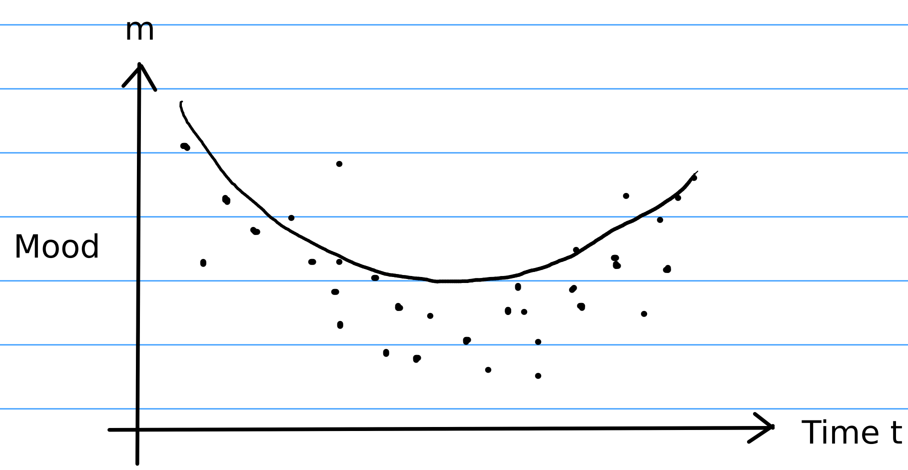
Model parameters ← (Linear) Model ← Output

Want → Model parameters

Independent Variables → (Linear) Model

Have → Output

Can you give an example?



Number of days in polar vortex

Model: $\hat{m}(t) = \alpha t^2 + \beta t + \gamma$

ers

by 2 will result in the predicted output being multiplied by 2.

How do we find the parameters then?

Have: 300 data points $(t_1, m_1), (t_2, m_2), \dots, (t_{300}, m_{300})$

Want: 3 unkowns (α, β, γ)

Write down equations:

$$m_1 \overset{!}{=} \hat{m}(t_1) = \alpha t_1^2 + \beta t_1 + \gamma 1$$

$$m_2 \overset{!}{=} \hat{m}(t_2) = \alpha t_2^2 + \beta t_2 + \gamma 1$$

$$\vdots$$

$$m_{300} \overset{!}{=} \hat{m}(t_{300}) = \alpha t_{300}^2 + \beta t_{300} + \gamma 1$$

$$\rightsquigarrow \begin{pmatrix} t_1^2 & t_1 & 1 \\ t_2^2 & t_2 & 1 \\ \vdots & \vdots & \vdots \\ t_{300}^2 & t_{300} & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \cong \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_{300} \end{pmatrix}$$

Go to matrix form

Write as least squares problem

Now solvable using QR.

Demo: Data Fitting