Fluid Dynamics Simulation of Rayleigh-Taylor Instability

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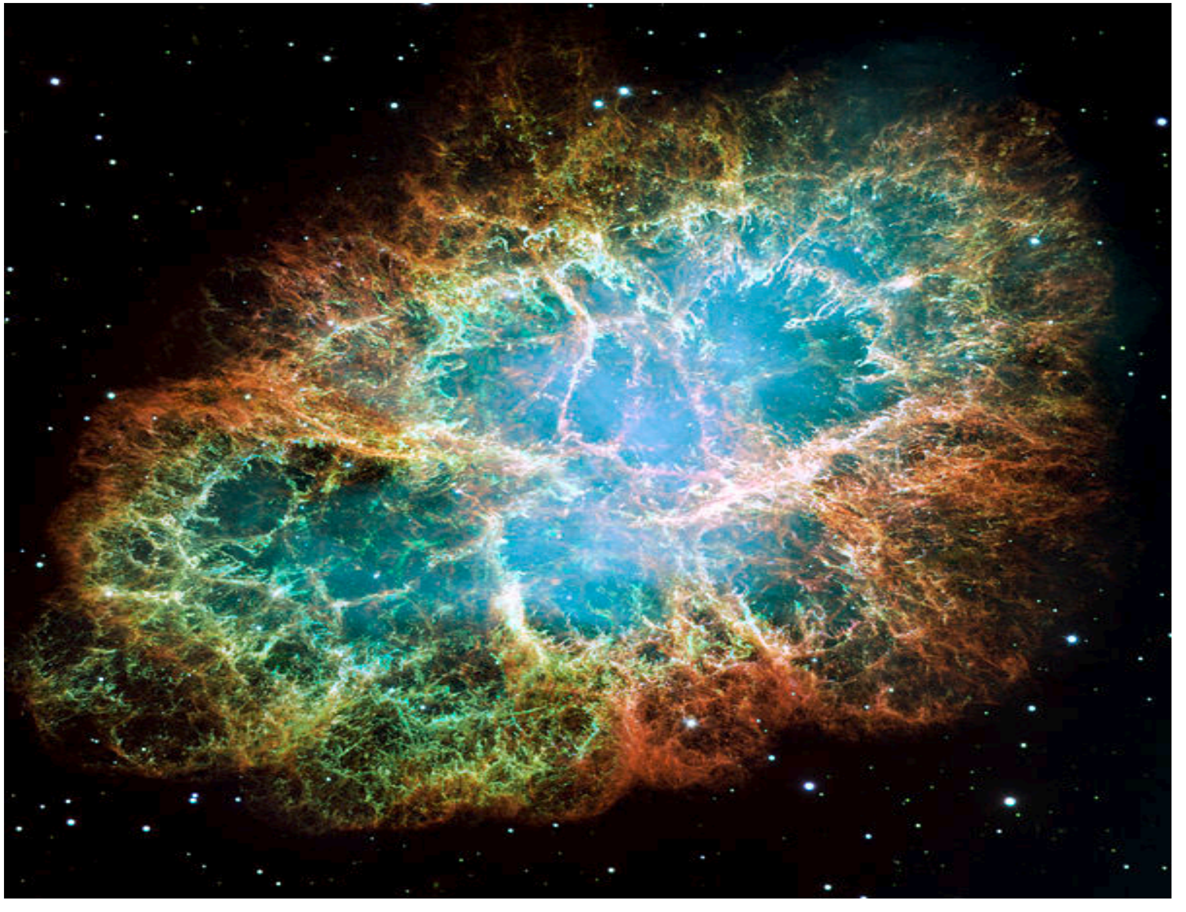
Rayleigh-Taylor Instability

- Introduction
  - Instability of an interface between two fluids of different densities
  - Instability is initialized by pertubations
- Cause
  - The dense fluid is pushed by the dilute fluid
  - Both fluids are subject to the gravity. The dense fluid is placed on top of the dilute fluid

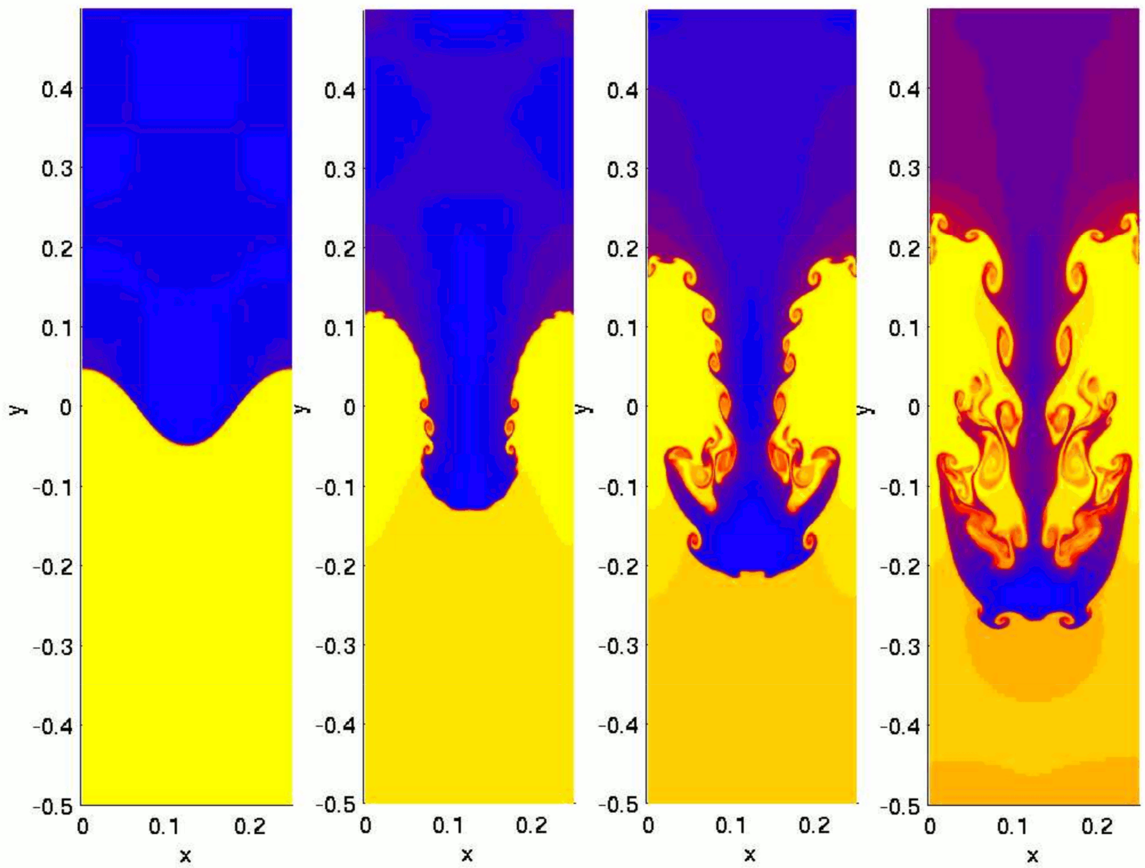
Rayleigh-Taylor Instability

- RT instability evident in Crab Nebula



Rayleigh-Taylor Instability

- RT fingers



Euler Equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0$$

Conservation of Mass

$$\frac{\partial \rho \vec{u}}{\partial t} + \nabla \cdot (\vec{u} \otimes (\rho \vec{u})) + \nabla p = \vec{0}$$

Conservation of Momentum

$$\frac{\partial E}{\partial t} + \nabla \cdot (\vec{u}(E + p)) = 0$$

Conservation of Energy

$$\vec{u} = (u, v, w)$$

$$E = \rho e + \frac{1}{2} \rho (u^2 + v^2 + w^2)$$

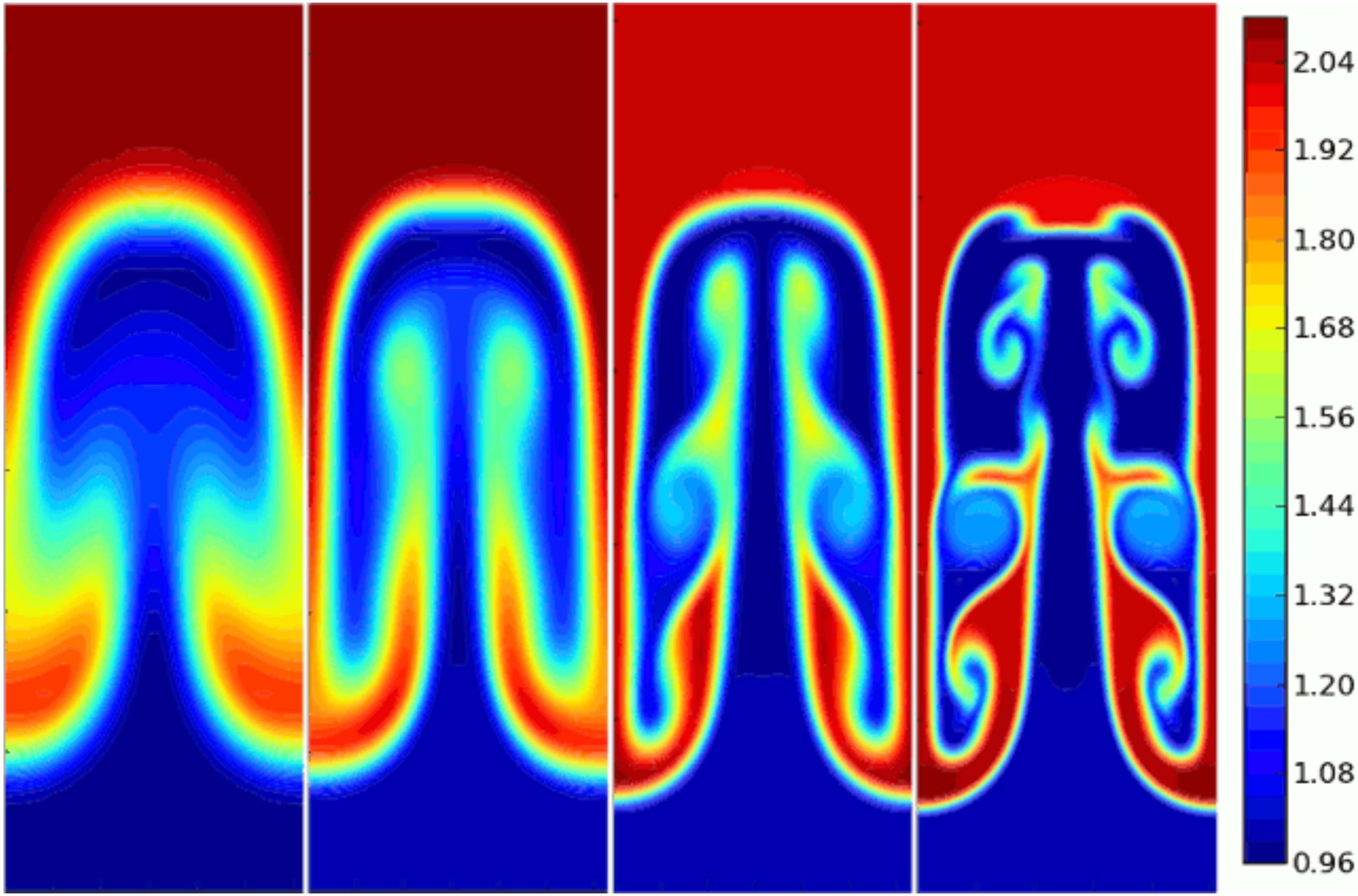
Euler Equations

Rewrite Euler equations in conservative form

$$\frac{\partial \vec{w}}{\partial t} + \frac{\partial \vec{f}_x}{\partial x} + \frac{\partial \vec{f}_y}{\partial y} + \frac{\partial \vec{f}_z}{\partial z} = 0$$

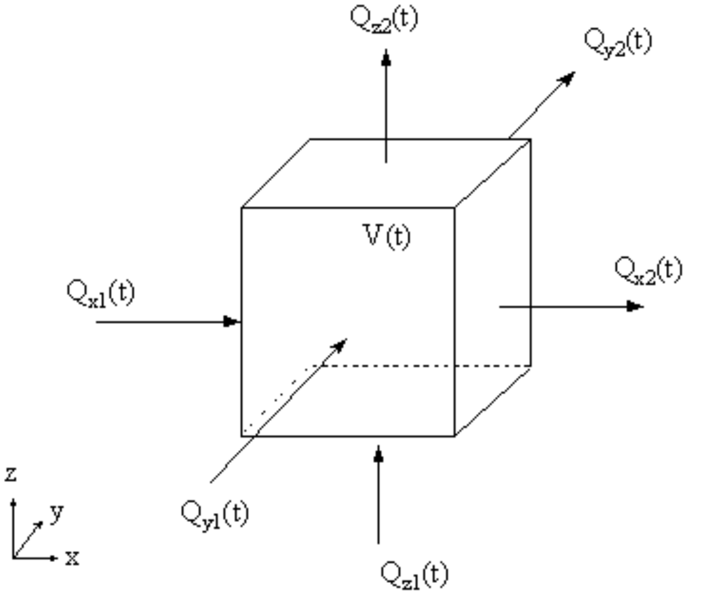
$$\vec{m} = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E \end{pmatrix} \quad \vec{f}_x = \begin{pmatrix} \rho u \\ p + \rho u^2 \\ \rho u v \\ \rho u w \\ u(E+p) \end{pmatrix} \quad \vec{f}_y = \begin{pmatrix} \rho v \\ \rho u v \\ p + \rho v^2 \\ \rho v w \\ v(E+p) \end{pmatrix} \quad \vec{f}_z = \begin{pmatrix} \rho w \\ \rho u w \\ \rho v w \\ p + \rho w^2 \\ w(E+p) \end{pmatrix}$$

Simulation Result



PDE Solver

- Introduction
  - Finite Difference Method
  - Finite Element Method
  - Finite Volume Method



1D Flux Computation: Riemann Problem

- In 1D, integrate $\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}(\boldsymbol{U})}{\partial x} = 0$ over space (Δx) and time (Δt)

$$\bar{u}_i(t) = \frac{1}{\Delta x_i} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x,t)\,dx$$

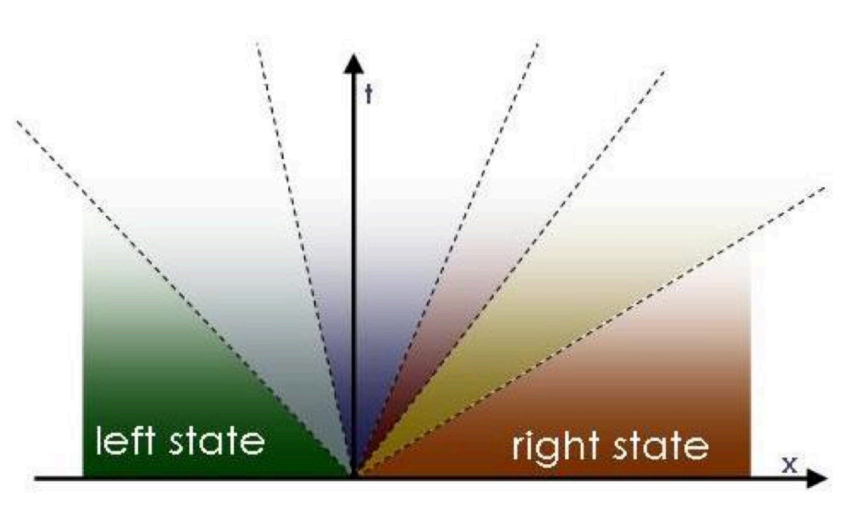
$$\bar{u}_i^{n+1} = \bar{u}_i^n - \frac{\Delta t}{\Delta x}\left(\tilde{F}_{i+\frac{1}{2}} - \tilde{F}_{i-\frac{1}{2}}\right)$$

$$\tilde{F}_{i+\frac{1}{2}} = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} F(u(x_{i+\frac{1}{2}}, t))\,dt$$

- Computation of the flux requires the (exact or approximate) solution of the Riemann problem at zone edges;
- *Riemann Problem*: given left and right states at a zone edge

$$\boldsymbol{U}(x, t=0) = \begin{cases} \boldsymbol{U}_L & \text{for} \quad x < 0 \\ \boldsymbol{U}_R & \text{for} \quad x > 0 \end{cases}$$

what is $\boldsymbol{U}(x,t)$?



- answer: the solution depends on the form of the conservation law.

Approximate Riemann Solver

- HLL(Harten-Lax-Van Leer) riemann solver

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial t} + \frac{\partial G}{\partial t} = 0$$

$$\frac{dU_{i,j}}{dt} = L(U) = -\frac{F_{i+1/2,j} - F_{i-1/2,j}}{\Delta x} - \frac{G_{i,j+1/2} - G_{i,j-1/2}}{\Delta y}$$

Flux Calculation

$$F^{HLL} = \frac{\alpha^{+}F^{L} + \alpha^{-}F^{R} - \alpha^{+}\alpha^{-}(U^{R} - U^{L})}{\alpha^{+} + \alpha^{-}}$$

$$\alpha^{\pm} = MAX\{0, \pm\lambda^{\pm}(U^{L}), \pm\lambda^{\pm}(U^{R})\}$$

$$\lambda^{\pm} = \upsilon \pm c_{s}$$

$$\Delta t < \Delta x / MAX(\alpha^{\pm}) \qquad c_{s} = \sqrt{\gamma P / \rho}$$

High Resolution Schemes

- High-order in time (Runge-Kutta)

$$U^{(1)} = U^n + \Delta t L(U^n) \qquad U^{(2)} = \frac{3}{4}U^n + \frac{1}{4}U^{(1)} + \frac{1}{4}\Delta t L(U^{(1)})$$

$$U^{n+1} = \frac{1}{3}U^n + \frac{2}{3}U^{(2)} + \frac{2}{3}\Delta t L(U^{(2)})$$

High-order in space(PLM)

$$c_{i+1/2}^{R} = c_{i+1} + 0.5 \min\text{mod}(\theta(c_{i+1} - c_i), 0.5(c_{i+2} - c_i), \theta(c_{i+2} - c_{i+1}))$$

$$c_{i+1/2}^{L} = c_i - 0.5 \min\text{mod}(\theta(c_i - c_{i-1}), 0.5(c_{i+1} - c_{i-1}), \theta(c_{i+1} - c_i))$$

Flux Limiter

- Avoid the spurious oscillations in HRS

$$\min\bmod(x,y,z) =$$

$$\frac{1}{4}|\operatorname{sgn}(x)+\operatorname{sgn}(y)|(\operatorname{sgn}(x)+\operatorname{sgn}(z))\min(|x|,|y|,|z|)$$

Demo