This short note describes the uniqueness proof for the exterior Helmholtz Dirichlet problem using the Brakhage-Werner/Panich/... representation

$$u = D\varphi - i\,\alpha S\varphi.$$

For simplicity, we'll choose the the scaling parameter $\alpha = 1$, so that

$$u = D\varphi + i\,S\varphi.$$

The exterior Dirichlet BC yields the integral equation (by way of the jump relations for S and D):

$$\frac{\varphi}{2} + D\varphi - i\,S\varphi = g.$$

Suppose $\varphi/2 + D\varphi - i\,S\varphi = 0$. We want to show $\varphi = 0$.

From the IE, we conclude that $\lim_{+} u = 0$. Using the 'exterior uniqueness helper' from class, we conclude that $u = 0$ in the entire exterior, thus $\lim_{+} \hat{n} \cdot \nabla u = 0$ also. The jump relations for the double and single layer then give us

$$\begin{aligned} 0 - (\hat{n} \cdot \nabla u)^{-} = [\hat{n} \cdot \nabla u] &= [\hat{n} \cdot \nabla\,(D\,\varphi - i\,S\varphi)] = -[i\,S'\varphi] = i\,\varphi \\ 0 - u^{-} = u^{+} - u^{-} = [u] &= [D\,\varphi - i\,S\varphi] = [D\varphi] = \varphi \end{aligned}$$

Equating right the right hand sides, we get

$$-i(\hat{n} \cdot \nabla u)^{-} = u^{-}.$$

Green's first theorem then yields

$$\int_{\Omega} -k^2|u|^2 + |\nabla u|^2 = \int_{\Omega} u \triangle \bar{u} + |\nabla u|^2 = \int_{\partial\Omega} u^{-} \overline{(\hat{n} \cdot \nabla u)^{-}} \mathrm{d}s = -i \int_{\partial\Omega} |u^{-}|^2 \mathrm{d}s.$$

Taking the imaginary part yields

$$\int_{\partial\Omega} |u^{-}|^2 \mathrm{d}s = 0.$$

Using $u^{+} = u^{-} = 0$ and the jump relation for the double layer, we obtain $\varphi = 0$ as desired.