The general jump relation for the double-layer potential is

$$D^{\pm}\varphi(x) = \mathrm{PV}\int_{\partial\Omega} (\hat{n}\cdot\nabla_y G(x,y))\varphi(y)\mathrm{d}y \pm \frac{1}{2}\varphi$$

for $x \in \partial\Omega$. We will show that for $\Omega \subseteq \mathbb{R}^3$ and $\varphi = 1$, i.e. what we have to show is

$$D^{\pm}1 = \mathrm{PV}\int_{\partial\Omega} \left(\hat{n}\cdot\nabla_y \frac{1}{4\pi|x-y|}\right)\mathrm{d}y \pm \frac{1}{2}.$$

Let's focus on the interior side of this like we did in the lecture, so we would like to show

$$D^{-}1 = \mathrm{PV}\int_{\partial\Omega} \left(\hat{n}\cdot\nabla_y \frac{1}{4\pi|x-y|}\right)\mathrm{d}y - \frac{1}{2}.$$

We can find the interior limit $D^{-}1$ from Green's formula for $x \in \Omega$:

$$(S(\hat{n}\cdot\nabla u))(x) - (D\,u)(x) = u(x)$$

by setting $u = 1$. Accordingly, $D^{-}1 = -1$.

As a result, what we're trying to show is

$$-\frac{1}{2} = \mathrm{PV}\int_{\partial\Omega} \left(\hat{n}\cdot\nabla_y \frac{1}{4\pi|x-y|}\right)\mathrm{d}y.$$

Proof:

Assume that $\partial\Omega$ is "smooth" (i.e. no corners, kinks, etc. allowed). Choose a target point $x \in \partial\Omega$ and consider a ball $B(x,r)$. Next, define

$$H_r := \Omega \cap \partial B(x,r),$$

which is the "half" of the sphere that intersects with Ω. As $r \to 0$, H_r becomes closer and closer to an actual half-sphere because of the smoothness of $\partial\Omega$.

If we let $\Gamma_r := \partial\Omega \setminus B(x,r) \cup H_r$ be the boundary of Ω with the radius r ball of x removed and the 'half-sphere' H_r substituted for that chunk of the boundary, we know that's a non-singular integral over the normal derivative of a harmonic function, so (by Green's second theorem)

$$\int_{\Gamma_r} \left(\hat{n}\cdot\nabla_y \frac{1}{4\pi|x-y|}\right)\mathrm{d}y = 0.$$

Splitting up Γ_r into its constituent parts, we obtain:

$$\begin{aligned}
0 &= \int_{\Gamma_r} \left(\hat{n}\cdot\nabla_y \frac{1}{4\pi|x-y|}\right)\mathrm{d}y \\
&= \int_{H_r} \left(\hat{n}\cdot\nabla_y \frac{1}{4\pi|x-y|}\right)\mathrm{d}y + \int_{\partial\Omega\setminus B(x,r)} \left(\hat{n}\cdot\nabla_y \frac{1}{4\pi|x-y|}\right)\mathrm{d}y \\
&= \frac{1}{4\pi}\int_{H_r} \left(\hat{n}\cdot\frac{x-y}{|x-y|^3}\right)\mathrm{d}y + \int_{\partial\Omega\setminus B(x,r)} \left(\hat{n}\cdot\nabla_y \frac{1}{4\pi|x-y|}\right)\mathrm{d}y \\
&= \frac{1}{4\pi}\int_{H_r} \left(\hat{n}\cdot\frac{\hat{n}r}{r^3}\right)\mathrm{d}y + \int_{\partial\Omega\setminus B(x,r)} \left(\hat{n}\cdot\nabla_y \frac{1}{4\pi|x-y|}\right)\mathrm{d}y \\
&= \frac{1}{4\pi r^2}\int_{H_r} \mathrm{d}y + \int_{\partial\Omega\setminus B(x,r)} \left(\hat{n}\cdot\nabla_y \frac{1}{4\pi|x-y|}\right)\mathrm{d}y
\end{aligned}$$

As we let $r \to 0$, we obtain

$$\begin{aligned} 0 &= \frac{\frac{1}{2}4\pi r^2}{4\pi r^2} + \mathrm{PV}\int_{\partial\Omega}\left(\hat{n}\cdot\nabla_y\frac{1}{4\pi|x-y|}\right)\mathrm{d}y \\ &= \frac{1}{2} + \mathrm{PV}\int_{\partial\Omega}\left(\hat{n}\cdot\nabla_y\frac{1}{4\pi|x-y|}\right)\mathrm{d}y, \end{aligned}$$

which shows our claim.