Homework Set 5

Problem 1: Multiply connected BVPs

In this problem, we will learn about a common issue which may cause integral equations to lose uniqueness: *connectedness of the domain.*

Recall how we handled the spurious nullspace in the integral equation using a double-layer representation for the *exterior* Dirichlet problem. We first showed that

$$\dim N(I/2 + D) = 1$$

and then threw a point charge into the domain to gain an extra degree of freedom which ended up squashing the nullspace.

It turns out that the *interior* Dirichlet problem can run into issues as well that can be addressed in a similar fashion.
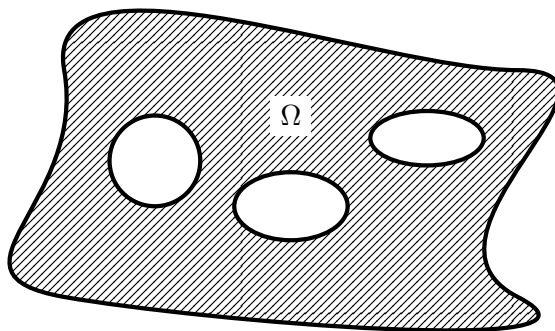

Figure 1: A multiply connected domain Ω with three 'inclusions' in the shapes of ellipses.

Assume that we are using a double-layer representation to solve a Laplace Dirichlet problem on a multiply connected interior domain Ω. (see Figure 1 for an illustration)

a) Extend our proof of
$$\dim N(I/2 - D) = 0,$$
(which was valid only for simply connected domains) to show that
$$\dim N(I/2 - D) = p,$$
where p is the number of bounded connected components of $\mathbb{R}^n \setminus \Omega$.

Terminology hint: The 'bounded connected components' of $\mathbb{R}^n \setminus \Omega$ in Figure 1 are exactly the ellipses. Realize that the remaining connected component (the 'proper exterior' of Ω, so to speak) is unbounded.
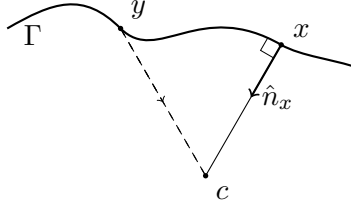
Figure 2: Geometric situation of 'Quadrature by Expansion'.

b) Establish existence of a solution to the interior Dirichlet boundary value problem

$$\triangle u = 0, \qquad u|_{\partial\Omega} = g$$

for multiply connected Ω through a modification of the 'usual' integral equation $I/2 - D = g$ analogous to what we used for the exterior Dirichlet problem, that is, by adding a 'point source' in each of the bounded connected components of $\mathbb{R}^n \setminus \Omega$.

For simplicity, only work this out for the three-dimensional case.

[patterned after Kress LIE, problem 6.3]

Turn in your answer in a PDF called `problem-1.pdf`.

Problem 2: A 2D BVP solver using 'Quadrature by Expansion'

This problem introduces you to a way of computing singular integrals called 'Quadrature by Expansion' or 'QBX'. For now, we will simply see the scheme at work, and we will learn more of the details later in the class.

To compute an integral such as the principal value of the double-layer potential at a (fixed, for now) target point x

$$\phi(x) := D\sigma(x) = \int_\Gamma \hat{n}_y \cdot \nabla_y G(x, y)\sigma(y)dy,$$

QBX exploits the smoothness of the potential ϕ away from (and right up to) Γ. To approximate the one-sided limit $\lim_{\xi \to x_+} \phi(\xi)$, QBX chooses a distance r from the curve and places a point c a distance r away from the curve along the target surface normal $\hat{n}_x$. We'll sometimes call c the *expansion center*. See Figure 2 for an illustration of the situation.

QBX then considers the (one-dimensional) function

$$f(t) = \phi(x + r(1-t)\hat{n}_x)$$

Realize that $f(1) = \phi(x)$ (which is the value we want) and $f(0) = \phi(c)$. It then simply uses Taylor's theorem

$$\phi(x) = f(1) \approx \sum_{i=0}^{p} f^{(i)}(0)\frac{1^i}{i!}$$

to obtain an approximate value for $\phi(x)$. Further, the Taylor coefficients

$$f^{(i)}(0) = \frac{\partial^i}{\partial t^i} \int_\Gamma \hat{n}_y \cdot \nabla_y G(x + r(1-t)\hat{n}_x, y)\sigma(y)dy, \Bigg|_{t=0}$$

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are non-singular integrals that we can simply compute by (regular) Gaussian quadrature.

The interior limit $\lim_{\xi \to x_-} \phi(\xi)$ can be found by a similar process on the opposite side of the boundary, and finally the principal ('on-surface') value can be computed by taking the average of the interior and exterior limits.

Use the following parameters for this problem:

- 8 nodes per subinterval (maximum polynomial degree 7),

- QBX order $p = 3$,

- the curve for which you computed the double-layer potential on homework 1,

$$\gamma(t) = \begin{pmatrix} \frac{3}{4}\cos(t - \pi/4)(1 + \sin(2t)/2) \\ \sin(t - \pi/4)(1 + \sin(2t)/2) \end{pmatrix},$$

- expansion center distance on the ith subinterval $r_i = h_i/2$, where

$$h_i = \int_{t_i}^{t_{i+1}} \|\gamma'(t)\| dt$$

is the length of the ith subinterval.

Do the following:

a) Use a computer algebra program of your choice to derive the Taylor expansion of the function f about zero for both the exterior and exterior limit.

Submit a transcript of your computer algebra session in PDF form as `expansion.pdf`.

Hint: See the last page of this assignment if you don't already have a favorite computer algebra system.

Hint 2: It's probably easiest to let the computer algebra system handle the translation from formula into code (and then copy and paste the result). For Maxima, the function `fortran(...)` does a reasonable job (even for Python).

b) Use the expansion formulas from the previous problem to implement a subroutine that evaluates the principal ('on-surface') value of the double-layer potential on the given curve.

Write a test script `constant-density.py` that takes two numbers `n1` and `n2` as command line arguments and computes the layer potential for a constant density on discretizations of the curve with `n1` and `n2` subintervals equispaced in the curve parameter t. It should then compute the L^2 error vs the (known!) value of the double-layer potential with a constant density for each discretization and further compute the empirical order of accuracy achieved by the calculation.

Hint: As described, the order of accuracy of the scheme should be p.

Hint: As a debugging aid, recall that the interior and exterior limits are known as well.

c) Now use the double-layer evaluator from the previous problem to implement a solver for the interior Dirichlet boundary value problem.

Write a test script `bvp.py` that takes two numbers `n1` and `n2` as command line arguments and solves the appropriate integral equation on discretizations of the curve with `n1` and `n2`

subintervals equispaced in the curve parameter t. Use GMRES to solve the resulting linear system. You should be able to choose residual tolerances in GMRES as small as 10^{-13}.

As (input) boundary data, use a fundamental solution centered at the point $(2,1)$ (which is outside of Ω). Realize that since the fundamental solution is harmonic (i.e. solves the PDE) away from $(2,1)$, this potential will also be the solution to our boundary value problem, by uniqueness. We will therefore call this potential the 'reference solution'.

From solving the integral equation, you obtain a density. Evaluate your representation with the computed density at the points
$$(\pm 0.2, 0), \quad (0, \pm 0.2),$$
which lie inside Ω.

It should then compute the l^2 (Note: lower-case l!) error in the potential at these points (vs. the reference solution) for the integral equation solutions on both discretizations and compute the empirical order of convergence of the method.

Your code should also print the number of iterations taken in GMRES in each case.

Implementation Hints:

- It will likely be expedient to use (and perhaps modify) your Legendre discretization toolkit from Homework 4 for use with this homework. Note that you will no longer need the indefinite integration functionality.

  Realize that the toolkit is suitable for curve representation nearly as-is, since curves are easily represented by two parametrization functions on a interval, e.g.

  $$\gamma(t) = (x(t), y(t)) = (\cos(2\pi t), \sin(2\pi t)), \qquad \text{for } t \in [0,1)$$

  for describing a circle, and the toolkit is already capable of describing functions on an interval.

- You can likely recycle some of your work from Homework Set 1, especially the formula for the curve parametrization and its derivatives.

Computer Algebra Software Hints

If you do not have a preferred computer algebra system, the following steps should allow you to install 'wxMaxima' on the class virtual machine to use for this assignment:

1. Double-click on 'Install and Remove Software' on the desktop. (Or use 'Applications / System / Synaptic Package Manager' to get the same program via the menu.)

2. Enter the super-secret password. (It's '`abc123`'.)

3. Click 'Reload' in the toolbar of the package manager. (Make sure your computer and the virtual machine have access to the internet. If something is not working, restart the VM.)

4. Using the 'Search' button in the toolbar (the rightmost button), search for '`wxmaxima`'.

5. Click the little square in front of the listing entry for `wxmaxima`, and click 'Mark for installation'.

6. Click 'Apply' in the toolbar, and confirm.

7. wxMaxima should now be available under 'Applications / Education'.

Hint: Maxima documentation is at

> `http://maxima.sourceforge.net/docs/manual/en/maxima.html`

Hint 2: You can generate a PDF from wxMaxima by printing to a PostScript file (`.ps`) and using `pdf2ps expansion.ps`.
`pdf2ps` is installed in the VM.