We want to solve the interior Neumann problem

$$\begin{aligned} \triangle u &= 0, \\ \hat{n} \cdot \nabla u &= g \quad \text{on} \quad \partial\Omega \end{aligned}$$

using an integral equation.

The solution of the BVP is not unique because if u is a solution, so is $u + c$. The naive way of doing so will use a single-layer representation

$$u(x) = S\sigma(x),$$

leading to the integral equation

$$S'\sigma + \frac{\sigma}{2} = g. \tag{1}$$

We'll bring in some auxiliary results.

Theorem 1. *(Kress LIE, Thm. 6.20)*

- $N(I/2 + D) = \operatorname{span}\{1\}$,
- $N(I/2 + S') = \operatorname{span}\{\psi\}$, *where the only thing we know about* ψ *is that* $\int_{\partial\Omega} \psi \neq 0$.

Theorem 2. *(Fredholm alternative, Kress LIE Thm. 4.14)*

$$(I/2 + S')(X) = N(I/2 + D)^{\perp}.$$

From Theorem 2 we conclude that for (1) to be solvable, g must be in $N(I/2 + D)^{\perp} = \operatorname{span}\{1\}^{\perp}$, i.e. it must satisfy

$$0 = (1, g) = \int_{\partial\Omega} 1 \cdot g = \int_{\partial\Omega} g.$$

Instead of (1), suppose we solve the modified integral equation

$$\frac{\sigma(x)}{2} + \underbrace{S'\sigma(x) + \underbrace{\mathbf{1}(x) \int_{\partial\Omega} \sigma(y) \mathrm{d}y}_{(**)}}_{(*)} = g. \tag{2}$$

This is a **rank-one modification** of (1), because the part $(**)$ has a range of dimension one. In particular, $(*)$ is a finite-dimensional perturbation of S', which means it is still compact, and the Fredholm alternative still applies.

Next, we show that **(2) has no nullspace**. To show that, let $\sigma \in C(\partial\Omega)$, and suppose that

$$\frac{\sigma}{2} + S'\sigma + \mathbf{1}(1, \sigma) = 0. \tag{3}$$

(3) is equivalent to

$$\frac{\sigma}{2} + S'\sigma = -\mathbf{1}(1, \sigma).$$

From theorem 2, we know that the left-hand side and right-hand side in the above equation are orthogonal to each other for any σ. Therefore

$$\begin{aligned}\frac{\sigma}{2}+S'\sigma &= 0 \quad \text{and}\\ (1,\sigma) &= 0.\end{aligned}$$

Since $\sigma/2+S'\sigma=0$, we know that $\sigma\in N(I/2+S')=\operatorname{span}\{\psi\}$. Because $(1,\sigma)=\int\sigma=0$, σ can't be a multiple of ψ, so $\sigma=0$.

The Fredholm alternative therefore yields that (2) has full range, i.e. we have **existence and uniqueness for (2)**, i.e. for any g we can find a σ that solves (2).

Next, we show $\int_{\partial\Omega}\boldsymbol{\sigma}=\mathbf{0}$. We assumed that $\int g=0$. Let σ be the unique solution of (2). Then testing (2) with 1 leads to

$$\begin{aligned}0=(1,0) &= \left(1,\frac{\sigma}{2}+S'\sigma+1(1,\sigma)-g\right)\\ &= \underbrace{\left(1,\frac{\sigma}{2}+S'\sigma\right)}_{0}+(1,1)(1,\sigma)-\underbrace{(1,g)}_{0},\end{aligned}$$

so $(1,\sigma)=\int_{\partial\Omega}\sigma=0$. Therefore, σ also solves the unmodified IE (1) and therefore the BVP.