Integral Equations and Fast Algorithms
Lecture 1: Intro

CS 598AK · August 27, 2013

Today

About this class

Integral equations: what?

Outline

Course Goal

PDE BVP goes in.

Accurate solution comes out, quickly.

Course Goal

PDE BVP goes in.

Accurate solution comes out,

~~quickly~~ efficiently.

Course Outline

Part 1: Theory (~ 6)

- Functional Analysis recap
- A zoology of IEs
- Riesz-Schauder theory
- Basic potential theory

Part 2: Numerics (~ 6)

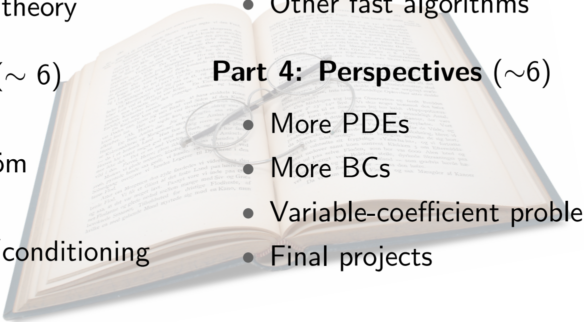
- Discretizations: Galerkin/Nyström
- Quadrature
- Linear systems/conditioning

Part 3: Algorithms ($\sim$6)

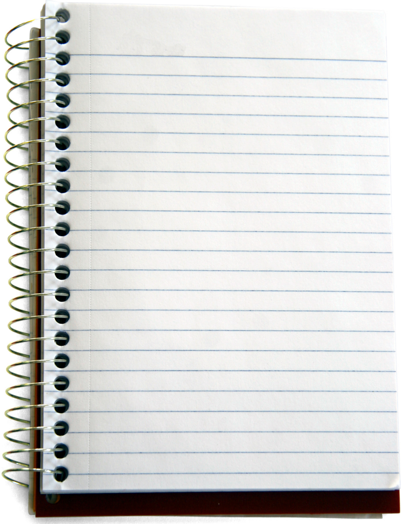
- 'Fast algorithm'?
- Fast Multipole
- FMM with Quadrature
- Other fast algorithms

Part 4: Perspectives ($\sim$6)

- More PDEs
- More BCs
- Variable-coefficient problems
- Final projects



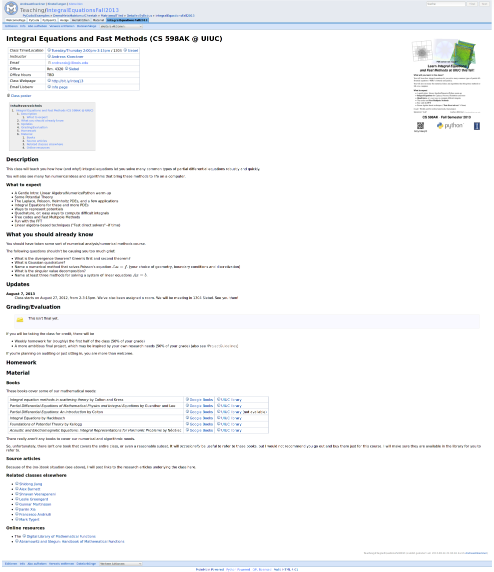
Sign-up sheet



Survey

- Home department
- Degree
- Longest program ever written?
  - in Python?
- Math preparation
  - Real analysis
  - Complex analysis
  - Functional analysis
- Written a PDE solver before?

Class web page



AndreasKloeckner | Einstellungen | Abmelden

Teaching/IntegralEquationsFall2013

PyCuda/Examples » DemoMetaMatrixmulCheetah » MatrixmulTiled » DetailedSyllabus » IntegralEquationsFall2013

WelcomePage | PyCuda | PyOpenCL | Hedge | HellsKitchen | Material | IntegralEquationsFall2013

Editieren | Info | Abo aufheben | Verweis entfernen | Dateianhänge | Weitere Aktionen:

Integral Equations and Fast Methods (CS 598AK @ UIUC)

Class Time/Location	Tuesday/Thursday 2:00pm-3:15pm / 1304 Siebel
Instructor	Andreas Kloeckner
Email	andreask@illinois.edu
Office	Rm. 4320 Siebel
Office Hours	TBD
Class Webpage	http://bit.ly/inteq13
Email Listserv	Info page

Class poster

Inhaltsverzeichnis

Description

This class will teach you how how (and why!) integral equations let you solve many common types of partial differential equations robustly and quickly.

You will also see many fun numerical ideas and algorithms that bring these methods to life on a computer.

What to expect

- A Gentle Intro: Linear Algebra/Numerics/Python warm-up
- Some Potential Theory
- The Laplace, Poisson, Helmholtz PDEs, and a few applications
- Integral Equations for these and more PDEs
- Ways to represent potentials
- Quadrature, or: easy ways to compute difficult integrals
- Tree codes and Fast Multipole Methods
- Fun with the FFT
- Linear algebra-based techniques ("Fast direct solvers"--if time)

What you should already know

You should have taken some sort of numerical analysis/numerical methods course.

The following questions shouldn't be causing you too much grief:

- What is the divergence theorem? Green's first and second theorem?
- What is Gaussian quadrature?
- Name a numerical method that solves Poisson's equation $\triangle u = f$. (your choice of geometry, boundary conditions and discretization)
- What is the singular value decomposition?
- Name at least three methods for solving a system of linear equations $Ax = b$.

Updates

August 7, 2013
Class starts on August 27, 2012, from 2-3:15pm. We've also been assigned a room. We will be meeting in 1304 Siebel. See you then!

Grading/Evaluation

This isn't final yet.

If you will be taking the class for credit, there will be

- Weekly homework for (roughly) the first half of the class (50% of your grade)
- A more ambitious final project, which may be inspired by your own research needs (50% of your grade) (also see /ProjectGuidelines)

If you're planning on auditing or just sitting in, you are more than welcome.

Homework

Material

Books

These books cover some of our mathematical needs:

Integral equation methods in scattering theory by Colton and Kress	Google Books	UIUC library
Partial Differential Equations of Mathematical Physics and Integral Equations by Guenther and Lee	Google Books	UIUC library
Partial Differential Equations: An Introduction by Colton	Google Books	UIUC library (not available)
Integral Equations by Hackbusch	Google Books	UIUC library
Foundations of Potential Theory by Kellogg	Google Books	UIUC library
Acoustic and Electromagnetic Equations: Integral Representations for Harmonic Problems by Nédélec	Google Books	UIUC library

There really *aren't* any books to cover our numerical and algorithmic needs.

So, unfortunately, there isn't *one* book that covers the entire class, or even a reasonable subset. It will *occasionally* be useful to refer to these books, but I would not recommend you go out and buy them just for this course. I will make sure they are available in the library for you to refer to.

Source articles

Because of the (no-)book situation (see above), I will post links to the research articles underlying the class here.

Related classes elsewhere

- Shidong Jiang
- Alex Barnett
- Shravan Veerapaneni
- Leslie Greengard
- Gunnar Martinsson
- Jianlin Xia
- Francesco Andriulli
- Mark Tygert

Online resources

- The Digital Library of Mathematical Functions
- Abramowitz and Stegun: Handbook of Mathematical Functions

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bit.ly/inteq13

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Online resources

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bit.ly/integ13

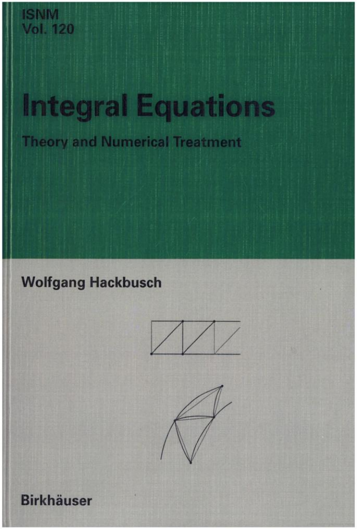
Posted: Virtual machine image (instructions in HW1)

Posted: Homework set 1
(Python, math/numerics warm-up, git, mechanics)
Due next week.

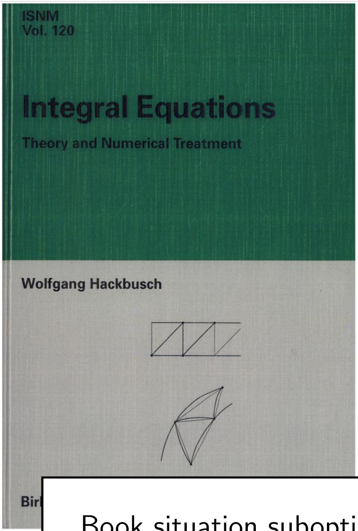
Listserv

inteq13@tiker.net

Books


ISNM
Vol. 120
Integral Equations
Theory and Numerical Treatment
Wolfgang Hackbusch
Birkhäuser

Books



Book situation suboptimal...

More Books

Grading

- 60% Homework
- 40% Final project

Smile! You're on camera

Lecture video will be posted soon after each class.

Questions?

?

Outline

Two specific elliptic PDEs

Laplace's Equation

$$\triangle u = 0$$

- Steady-state $\partial_t u = 0$ of wave propagation, heat conduction
- Electric potential u for applied voltage
- Minimal surfaces/"soap films"
- ∇u as velocity of incompressible flow

Two specific elliptic PDEs

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Helmholtz Equation

$$\triangle u + k^2 u = 0$$

- Assume time-harmonic behavior $\tilde{u} = e^{\pm i\omega t} u(x)$ in time-domain wave equation:

$$\partial_t^2 \tilde{u} = \triangle \tilde{u}$$

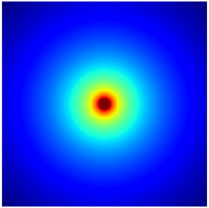
- Sign in $\tilde{u}$ determines direction of wave:
  - Incoming/outgoing if free-space problem

Applications: Propagation of sound, electromagnetic waves

Fundamental Solutions

Laplace Equation

$-\triangle u = \delta$

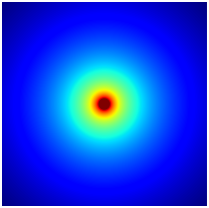


Monopole

Fundamental Solutions

Laplace Equation

$-\triangle u = \delta$



Monopole

Helmholtz Equation

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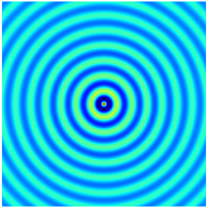

Monopole

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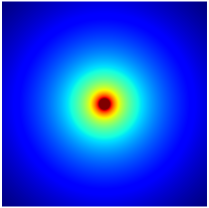
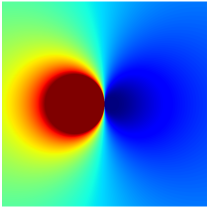


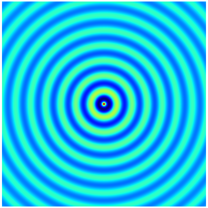
Monopole



Dipole

Helmholtz Equation

$\triangle u + k^2 u = \delta$



Monopole

Fundamental Solutions

Laplace Equation

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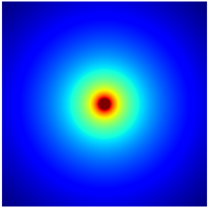

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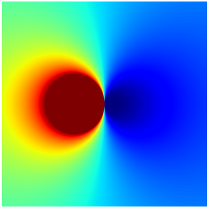

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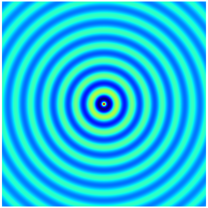

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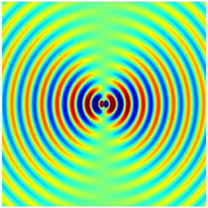

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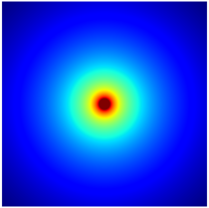

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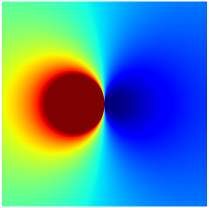
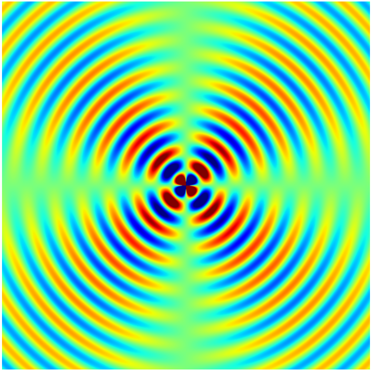


Dipole

Helmholtz Equation

Can take this arbitrarily far:



Quadrupole, ...

Fundamental Solutions

Laplace Equation

$-\triangle G = \delta$

Monopole:

$$G(x) = \begin{cases} \frac{1}{-2\pi} \log |x| & \text{2D} \\ \frac{1}{4\pi} \frac{1}{|x|} & \text{3D} \end{cases}$$

Fundamental Solutions

Laplace Equation

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Helmholtz Equation

$(\triangle + k^2) G = \delta$

Monopole:

$$G(x) = \begin{cases} \frac{i}{4} H_0^1(k|x|) & \text{2D} \\ \frac{1}{4\pi} \frac{e^{ik|x|}}{|x|} & \text{3D} \end{cases}$$

Fundamental Solutions

Laplace Equation

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Dipole:

$$\frac{\partial}{\partial_x} G(x)$$

Helmholtz Equation

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Dipole:

$$\frac{\partial}{\partial_x} G(x)$$

Building Solutions

Main question for numerical solution of PDEs:

How is the solution represented?

Our choice here: *Sums of fundamental solutions*

$$\tilde{u}(x) = \sum_{i=1}^{N} G(|x - y_i|)\sigma_i$$

located at *source points* y_i

Building Solutions

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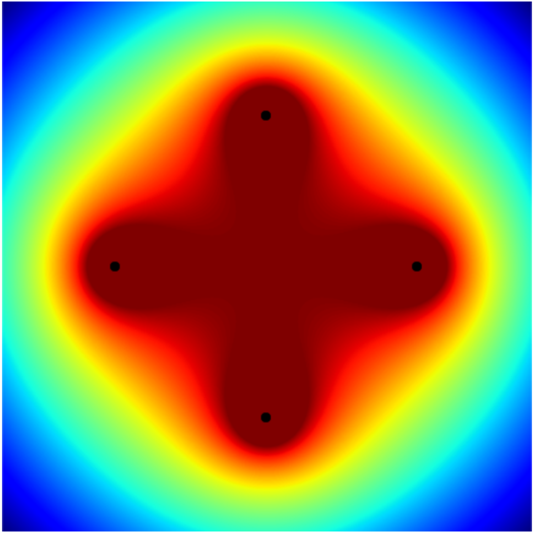
- Linearity → must satisfy PDE
- Boundary conditions: not necessarily
- Is the solution reachable in this way?
  - Uniqueness?

Summing Fundamental Solutions



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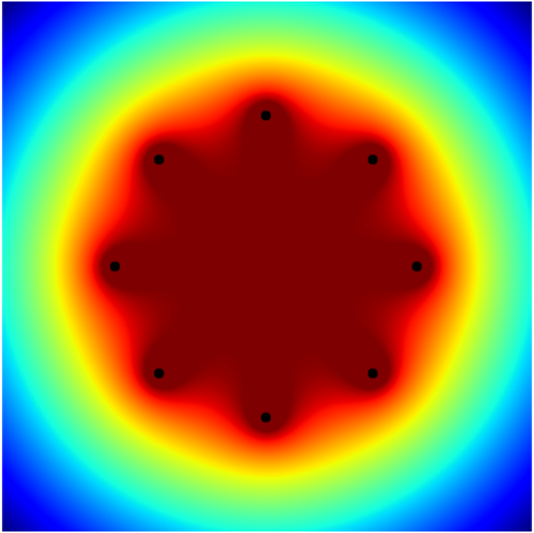

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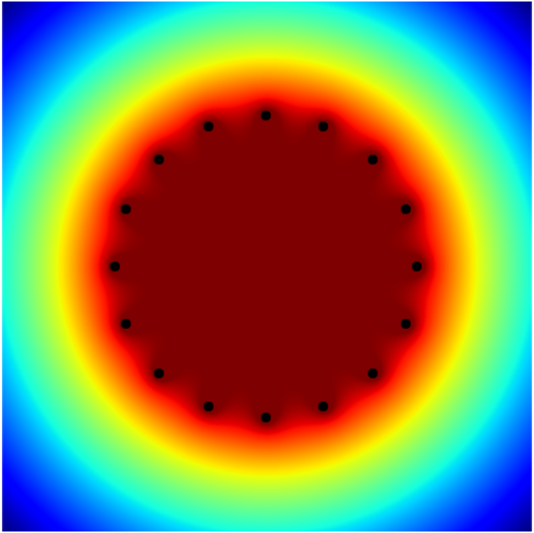

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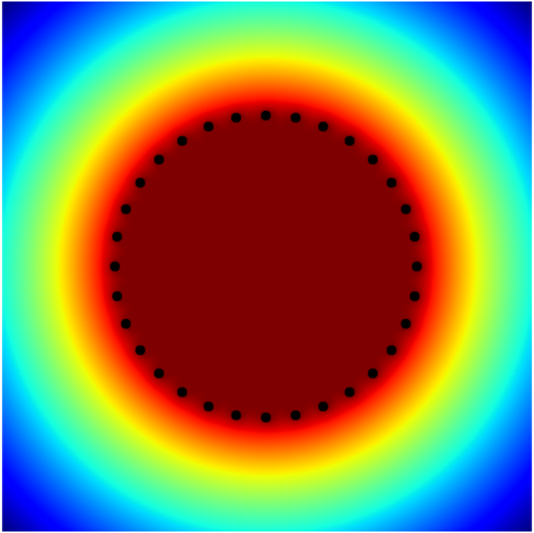

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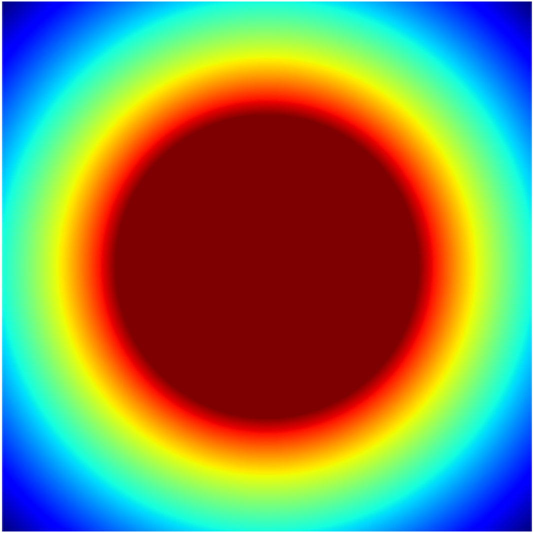

Layer Potentials

$$(S_k\sigma)(x) := \int_\Gamma G_k(x-y)\sigma(y)ds_y$$

$$(S'_k\sigma)(x) := n\cdot\nabla_x PV\int_\Gamma G_k(x-y)\sigma(y)ds_y$$

$$(D_k\sigma)(x) := PV\int_\Gamma n\cdot\nabla_y G_k(x-y)\sigma(y)ds_y$$

$$(D'_k\sigma)(x) := n\cdot\nabla_x f.p.\int_\Gamma n\cdot\nabla_y G_k(x-y)\sigma(y)ds_y$$

- Operators–map function σ on Γ to...
  - ...function on $\mathbb{R}^n$
  - ...function on Γ (in particular)
- S'' (and higher) analogously
- Called *layer potentials*
- G_k is the Helmholtz kernel ($k = 0 \to$ Laplace)

Layer potential demo time

Solving a BVP with integral equations

Solve a (interior Laplace Dirichlet) BVP, $\partial\Omega = \Gamma$

$$\triangle u = 0 \quad \text{in}\Omega, \qquad u|_\Gamma = f|_\Gamma.$$

1. Pick representation:

$$u(x) := (S\sigma)(x)$$

2. Take (interior) limit onto Γ:

$$u|_\Gamma = S\sigma$$

3. Enforce BC:

$$u|_\Gamma = f$$

4. Solve resulting linear system:

$$S\sigma = f$$

5. Obtain PDE solution in Ω by evaluating representation

BVP solve demo time

What to do?

1. Pick representation:

$$u(x) := (D\sigma)(x)$$

2. Take (interior) limit onto Γ:

$$u|_\Gamma = D\sigma - \sigma/2$$

3. Enforce BC:

$$u|_\Gamma = f$$

4. Solve resulting linear system:

$$(D - \operatorname{Id}/2)\sigma = f$$

5. Obtain PDE solution in Ω by evaluating representation

Wha[t to do?]

'Second-kind' integral equation
Previous one: 'First-kind'

1. Pick representation:

$$u(x) := (D\sigma)(x)$$

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Second-kind BVP solve demo time

Questions?

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- Notebook: sxc.hu/abeall
- Question mark: sxc.hu/svilen001
- Camera: sxc.hu/Kolobsek