Integral Equations and Fast Algorithms
Lecture 1: Intro

CS 598AK · August 27, 2013
Today

About this class

Integral equations: what?
Outline

About this class

Integral equations: what?
Course Goal

PDE BVP goes in.

Accurate solution comes out, quickly.
Course Goal

PDE BVP goes in.

Accurate solution comes out, quickly efficiently.
Course Outline

Part 1: Theory (∼ 6)
- Functional Analysis recap
- A zoology of IEs
- Riesz-Schauder theory
- Basic potential theory

Part 2: Numerics (∼ 6)
- Discretizations: Galerkin/Nyström
- Quadrature
- Linear systems/conditioning

Part 3: Algorithms (∼6)
- ‘Fast algorithm’?
- Fast Multipole
- FMM with Quadrature
- Other fast algorithms

Part 4: Perspectives (∼6)
- More PDEs
- More BCs
- Variable-coefficient problems
- Final projects
Sign-up sheet
Survey

- Home department
- Degree
- Longest program ever written?
  - in Python?
- Math preparation
  - Real analysis
  - Complex analysis
  - Functional analysis
- Written a PDE solver before?
Integral Equations and Fast Methods (CS 598AK @ UIUC)

Class: Mon, Wed, 1-3:30pm, 3512 E. Science Library
Instructor: Andreas Kuehlwein
Office: 1409 E. Science Library
Office Hours: TBD
Class webpage: bit.ly/inteq13

Description
This course will teach you how to use computers to solve partial differential equations numerically and quickly.
You will also see many fun numerical ideas and algorithms that bring these methods to life on a computer.

What to expect
- A gentle start: Linear Algebra and Linear Systems
- An overview of the history and applications of Integral Equations
- Methods for solving Integral Equations
- Applications to Boundary Value Problems
- Fast Methods for solving Integral Equations
- Advanced topics in Integral Equations

What you should already know
You should have taken an upper-level numerical methods course.

What a typical day would look like:
- Lecture: Mathematical concepts and algorithms
- Homework: Writing code to implement the algorithms
- Weekly test: Assessing your understanding of the material

Updates
August 20, 2013
Welcome to class and see you there!

Grading/Evaluation
You will have:
- A Midterm
- A Final
- Weekly homework

Homework
- Weekly homework
- Midterm
- Final

Material

Books

- Numerical Linear Algebra and Applications by Trefethen and Bau III
- Introduction to Integral Equations with Applications by Lakin
- Fast Iterative Solution of Linear Systems by Saad
- Integral Equations: Theory and Numerical Treatment
- Advanced Engineering Mathematics

Additional resources
- Class notes
- Lecture recordings

Source articles
- Related papers
- Research articles

Online resources
- Course website
- Class blog

bit.ly/inteq13
Class web page

Integral Equations and Fast Methods (CS 598AK @ UIUC)

Description
This class will teach you how to solve integral equations. We'll cover some of the most common types of integral equations and how to solve them numerically and analytically. You will also learn about fast methods for solving integral equations.

What to expect
• Weekly homework assignments
• Two project assignments
• Midterm and final exams

What you should already know
You should have taken some sort of mathematical methods course. If you have not, here are some suggestions:
• Understanding of linear algebra
• Knowledge of differential equations

Updates
August 1, 2013
Graduation/Evaluation
The final grade will be based on your performance in the following categories:

• Weekly homework assignments
• Two project assignments

Material
Books
These books cover some of our course material:

• Integral equations and their applications by Kress
• Partial differential equations by Evans
• Integral transforms by Kress
• Foundations of Potential Theory by Relle

Online resources
• The Digital Library of Mathematical Functions
• NIST Handbook of Mathematical Functions

Posted: Virtual machine image (instructions in HW1)

Posted: Homework set 1
(Python, math/numerics warm-up, git, mechanics)
Due next week.
Listserv

in.teq13@tiker.net
Books

Integral Equations
Theory and Numerical Treatment

Wolfgang Hackbusch

Birkhäuser
Books

Book situation suboptimal...
More Books

Oliver Dimon Kellogg
Foundations of Potential Theory
Grading

• 60% Homework
• 40% Final project
Smile! You’re on camera

Lecture video will be posted soon after each class.
Questions?
Outline

About this class

Integral equations: what?
Two specific elliptic PDEs

Laplace’s Equation

\[ \Delta u = 0 \]

- Steady-state \( \partial_t u = 0 \) of wave propagation, heat conduction
- Electric potential \( u \) for applied voltage
- Minimal surfaces/“soap films”
- \( \nabla u \) as velocity of incompressible flow

Helmholtz Equation

\[ \Delta u + k^2 u = 0 \]

- Assume time-harmonic behavior \( \tilde{u} = e^{\pm i\omega t} u(x) \) in time-domain wave equation:
  \[ \partial^2_t \tilde{u} = \Delta \tilde{u} \]
- Sign in \( \tilde{u} \) determines direction of wave:
  - Incoming/outgoing if free-space problem

Applications:
- Propagation of sound, electromagnetic waves
Two specific elliptic PDEs

Laplace’s Equation

\[ \triangle u = 0 \]

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  - Incoming/outgoing if free-space problem

Applications: Propagation of sound, electromagnetic waves
Laplace Equation

\[-\Delta u = \delta\]
Fundamental Solutions

Laplace Equation

\[-\Delta u = \delta\]

Monopole

Helmholtz Equation

\[\Delta u + k^2 u = \delta\]

Monopole

Can take this arbitrarily far:
Quadrupole, . . .
Fundamental Solutions

**Laplace Equation**

\[-\Delta u = \delta\]

Monopole

Dipole

**Helmholtz Equation**

\[\Delta u + k^2 u = \delta\]

Monopole

Can take this arbitrarily far: Quadrupole, ...
Fundamental Solutions

Laplace Equation
\[-\Delta u = \delta\]

- Monopole
- Dipole

Helmholtz Equation
\[\Delta u + k^2 u = \delta\]

- Monopole
- Dipole

Can take this arbitrarily far: Quadrupole, ...
Fundamental Solutions

Laplace Equation

\[-\triangle u = \delta\]

Monopole

Dipole

Helmholtz Equation

Can take this arbitrarily far:

Quadrupole, …
Fundamental Solutions

Laplace Equation

\[-\Delta G = \delta\]

Monopole:

\[G(x) = \begin{cases} 
\frac{1}{-2\pi} \log |x| & \text{2D} \\
\frac{1}{4\pi} \frac{1}{|x|} & \text{3D}
\end{cases}\]
<table>
<thead>
<tr>
<th>Equation Type</th>
<th>Equation</th>
<th>Monopole</th>
<th>Dipole</th>
</tr>
</thead>
<tbody>
<tr>
<td>Laplace Equation</td>
<td>$-\Delta G = \delta$</td>
<td>$G(x) = \begin{cases} \frac{1}{-2\pi} \log</td>
<td>x</td>
</tr>
<tr>
<td>Helmholtz Equation</td>
<td>$(\Delta + k^2) G = \delta$</td>
<td></td>
<td></td>
</tr>
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Fundamental Solutions

Laplace Equation

\[-\nabla^2 G = \delta\]

Monopole:

\[G(x) = \begin{cases} \frac{1}{-2\pi} \log |x| & 2D \\ \frac{1}{4\pi} \frac{1}{|x|} & 3D \end{cases}\]

Dipole:

\[\frac{\partial}{\partial x} G(x)\]

Helmholtz Equation

\[(\nabla^2 + k^2) G = \delta\]

Monopole:

\[G(x) = \begin{cases} i \frac{H^1_0(k|x|)}{4\pi} |x| & 2D \\ \frac{1}{4\pi} e^{ik|x|} & 3D \end{cases}\]

Dipole:

\[\frac{\partial}{\partial x} G(x)\]
Main question for numerical solution of PDEs:

**How is the solution represented?**

Our choice here: *Sums of fundamental solutions*

\[
\tilde{u}(x) = \sum_{i=1}^{N} G(|x - y_i|)\sigma_i
\]

located at *source points* \(y_i\)
Main question for numerical solution of PDEs:

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- Linearity → must satisfy PDE
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- Linearity → must satisfy PDE
- Boundary conditions: not necessarily
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- Linearity \( \rightarrow \) must satisfy PDE
- Boundary conditions: not necessarily
- Is the solution reachable in this way?
Main question for numerical solution of PDEs:

How is the solution represented?

Our choice here: *Sums of fundamental solutions*

\[ \tilde{u}(x) = \sum_{i=1}^{N} G(|x - y_i|)\sigma_i \]

located at *source points* \( y_i \)

- Linearity → must satisfy PDE
- Boundary conditions: not necessarily
- Is the solution reachable in this way?
  - Uniqueness?
Summing Fundamental Solutions
Summing Fundamental Solutions
Summing Fundamental Solutions
Summing Fundamental Solutions
Summing Fundamental Solutions
Layer Potentials

\[ (S_k \sigma)(x) := \int_{\Gamma} G_k(x - y) \sigma(y) \, ds_y \]

\[ (S'_k \sigma)(x) := n \cdot \nabla_x PV \int_{\Gamma} G_k(x - y) \sigma(y) \, ds_y \]

\[ (D_k \sigma)(x) := PV \int_{\Gamma} n \cdot \nabla_y G_k(x - y) \sigma(y) \, ds_y \]

\[ (D'_k \sigma)(x) := n \cdot \nabla_x f.p. \int_{\Gamma} n \cdot \nabla_y G_k(x - y) \sigma(y) \, ds_y \]

- Operators—map function \( \sigma \) on \( \Gamma \) to...
  - ...function on \( \mathbb{R}^n \)
  - ...function on \( \Gamma \) (in particular)

- \( S'' \) (and higher) analogously

- Called layer potentials

- \( G_k \) is the Helmholtz kernel (\( k = 0 \) → Laplace)
Layer potential demo time
Solving a BVP with integral equations

Solve a (interior Laplace Dirichlet) BVP, $\partial\Omega = \Gamma$

$$\Delta u = 0 \quad \text{in } \Omega, \quad u|_{\Gamma} = f|_{\Gamma}. $$

1. Pick representation:

$$u(x) := (S\sigma)(x)$$

2. Take (interior) limit onto $\Gamma$:

$$u|_{\Gamma} = S\sigma$$

3. Enforce BC:

$$u|_{\Gamma} = f$$

4. Solve resulting linear system:

$$S\sigma = f$$

5. Obtain PDE solution in $\Omega$ by evaluating representation
BVP solve demo time
What to do?

1. Pick representation:
   \[ u(x) := (D\sigma)(x) \]

2. Take (interior) limit onto \( \Gamma \):
   \[ u|_\Gamma = D\sigma - \sigma/2 \]

3. Enforce BC:
   \[ u|_\Gamma = f \]

4. Solve resulting linear system:
   \[ (D - \text{Id}/2)\sigma = f \]

5. Obtain PDE solution in \( \Omega \) by evaluating representation
What to do?

1. Pick representation:

\[ u(x) := (D\sigma)(x) \]

2. Take (interior) limit onto \( \Gamma \):

\[ u|_{\Gamma} = D\sigma - \sigma/2 \]

3. Enforce BC:

\[ u|_{\Gamma} = f \]

4. Solve resulting linear system:

\[ (D - \text{Id}/2)\sigma = f \]

5. Obtain PDE solution in \( \Omega \) by evaluating representation

‘Second-kind’ integral equation

Previous one: ‘First-kind’
Second-kind BVP solve demo time
Image Credits

- Notebook: sxc.hu/abeall
- Question mark: sxc.hu/svilen001
- Camera: sxc.hu/Kolobsek