Integral Equations and Fast Algorithms
Lecture 10: Spectral Theory, Potential Theory

CS 598AK · September 26, 2013

Early feedback

Today

Mini Spectral Theory

Harmonic Potential Theory

Outline

Spectral theory: terminology

$A : X \to X$ bounded, λ is a ______ value:

Definition (Eigenvalue)

There exists an element $\varphi \in X$, $\varphi \neq 0$ with $A\varphi = \lambda\varphi$.

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What's special about ∞-dim here?

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Definition (Spectrum)

$\sigma(A) := \mathbb{C} \setminus \rho(A)$

Spectral Theory of Compact Operators

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Rephrase last two: how many eigenvalues with $|\cdot| \geq R$?

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How might that relate to compactness?

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Intuition about Compact Operators: recap

- What do they do to high-frequency data?
- What do they do to low-frequency data?
- Don't confuse $I - A$ with A itself!
  (For example: $\dim N(A)$ vs $\dim N(I - A)$)

Outline

Recap: Laplace fundamental solution

Definition (Harmonic function)

$$\triangle u = 0$$

Fundamental solution:

$$G(x) = \begin{cases} \frac{1}{-2\pi} \log |x| & \text{2D} \\ \frac{1}{4\pi} \frac{1}{|x|} & \text{3D} \end{cases}$$

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Relationship to free-space Poisson equation?

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$$\triangle \operatorname{Re} f, \operatorname{Im} f = 0$$

i.e. harmonic for f *differentiable* in $\mathbb{C}$. (identifying $\mathbb{R}^2$ with $\mathbb{C}$)

Recap: Layer Potentials

$$(S\sigma)(x) := \int_\Gamma G(x-y)\sigma(y)ds_y$$

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Alternate ("standard") nomenclature:

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S	V
D	K
S'	K'
D'	T

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Harmonic–where?

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Is the double layer *actually* weakly singular?

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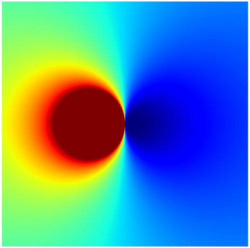
Definition (Weakly singular kernel)

- K defined, continuous everywhere except at $x = y$
- There exist $C > 0$, $\alpha \in (0, n-1]$ such that

$$|K(x,y)| \leq C|x-y|^{\alpha-n+1} \qquad (x, y \in \partial\Omega,\ x \neq y)$$



$$\frac{\partial}{\partial_x} log(|0-x|) = \frac{x}{x^2+y^2}$$

- Singularity with approach on $y = 0$?
- Singularity with approach on $x = 0$?

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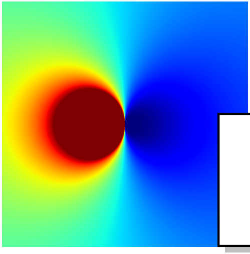

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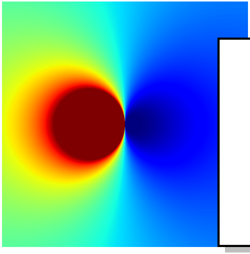

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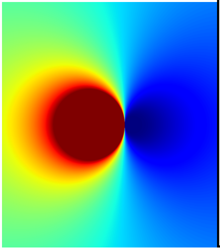

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Would like an analytical tool that requires 'less' fanciness.

Questions?

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