Early feedback
Today

Mini Spectral Theory

Harmonic Potential Theory
Outline

Mini Spectral Theory

Harmonic Potential Theory
Spectral theory: terminology

$A : X \to X$ bounded, $\lambda$ is a ______ value:

**Definition (Eigenvalue)**

There exists an element $\varphi \in X$, $\varphi \neq 0$ with $A\varphi = \lambda\varphi$. 
Spectral theory: terminology

$A : X \to X$ bounded, $\lambda$ is a ______ value:

**Definition (Eigenvalue)**

There exists an element $\varphi \in X$, $\varphi \neq 0$ with $A\varphi = \lambda\varphi$.

**Definition (Regular value)**

The “resolvent” $(\lambda I - A)^{-1}$ exists and is bounded.
Spectral theory: terminology

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Can a value be regular and “eigen” at the same time?
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What’s special about $\infty$-dim here?
Spectral theory: terminology

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<td>$\sigma(A) := \mathbb{C} \setminus \rho(A)$</td>
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Spectral Theory of Compact Operators

Theorem

\[ A : X \rightarrow X \text{ compact linear operator, } X \text{ } \infty\text{-dim.} \]

Then:

- \( 0 \in \sigma(A) \)

- \( \sigma(A) \{ 0 \} \) consists only of eigenvalues

- \( \sigma(A) \{ 0 \} \) is at most countable

- \( \sigma(A) \) has no accumulation point except for 0

Rephrase last two: how many eigenvalues with \( |\cdot| \geq R \)? How might that relate to compactness?
Theorem

\( A : X \to X \) compact linear operator, \( X \) \( \infty \)-dim.

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Let $A : X \to X$ be a compact linear operator, where $X$ is infinite-dimensional. Then:

- $0 \in \sigma(A)$ (show! Hint: $A^{-1}A$)
- $\sigma(A) \setminus \{0\}$ consists only of eigenvalues
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Rephrase last two: how many eigenvalues with $|\cdot| \geq R$?

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Intuition about Compact Operators: recap

- What do they do to high-frequency data?
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- What do they do to low-frequency data?
Intuition about Compact Operators: recap

- What do they do to high-frequency data?
- What do they do to low-frequency data?
- Don’t confuse $I - A$ with $A$ itself!
  (For example: $\dim N(A)$ vs $\dim N(I - A)$)
Outline

Mini Spectral Theory

Harmonic Potential Theory
Recap: Laplace fundamental solution

Definition (Harmonic function)

\[ \triangle u = 0 \]

Fundamental solution:

\[ G(x) = \begin{cases} \frac{1}{-2\pi} \log|x| & 2D \\ \frac{1}{4\pi} \frac{1}{|x|} & 3D \end{cases} \]

\[ -\triangle G(x) = \delta(x) \rightarrow \text{exact meaning?} \]
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\[ \triangle \text{Re } f, \text{Im } f = 0 \]

i.e. harmonic for \( f \) differentiable in \( \mathbb{C} \). (identifying \( \mathbb{R}^2 \) with \( \mathbb{C} \))
Recap: Layer Potentials

\[(S\sigma)(x) := \int_{\Gamma} G(x - y)\sigma(y)ds_y\]

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Harmonic–where?
On the double layer again

Is the double layer *actually* weakly singular?
On the double layer again

Is the double layer actually weakly singular?

**Definition (Weakly singular kernel)**

- $K$ defined, continuous everywhere except at $x = y$
- There exist $C > 0$, $\alpha \in (0, n - 1]$ such that
  \[ |K(x, y)| \leq C|x - y|^{\alpha-n+1} \quad (x, y \in \partial\Omega, x \neq y) \]

\[ \frac{\partial}{\partial x} \log(|0 - x|) = \frac{x}{x^2 + y^2} \]

- Singularity with approach on $y = 0$?
- Singularity with approach on $x = 0$?
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Would like an analytical tool that requires ‘less’ fanciness.
Questions?