Integral Equations and Fast Algorithms
Lecture 11: Potential Theory

CS 598AK · October 1, 2013

Feedback results

- Can we get to the numerics already?
  - Connect math to methods
  - Swamp of theorems
- Need homework on FA bits of the course.
- HW ok – want more HW
- More (EM?) examples
- Goals sometimes unclear
- Slide typos (I try to fix them before posting–let me know if I've missed something.)
- Books (I know)
  - Post book chapters
- Food coma
- Use Matlab instead

Today

Harmonic Potential Theory

Outline

Harmonic Potential Theory

On the double layer again

Is the double layer *actually* weakly singular?

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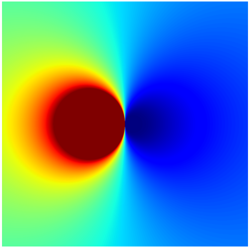
Definition (Weakly singular kernel)

- K defined, continuous everywhere except at $x = y$
- There exist $C > 0$, $\alpha \in (0, n-1]$ such that

$$|K(x,y)| \leq C|x-y|^{\alpha-n+1} \qquad (x, y \in \partial\Omega,\ x \neq y)$$



$$\frac{\partial}{\partial_x} log(|0-x|) = \frac{x}{x^2+y^2}$$

- Singularity with approach on $y = 0$?
- Singularity with approach on $x = 0$?

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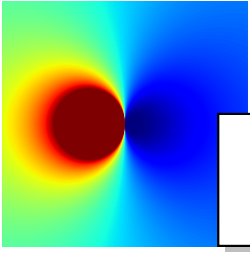

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So life is simultaneously worse and better than discussed.

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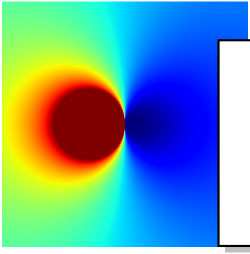

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Is S' compact?

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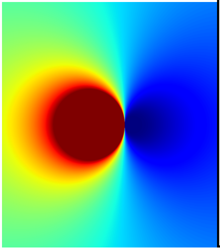

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How about 3D? $(-x/|x|^3)$

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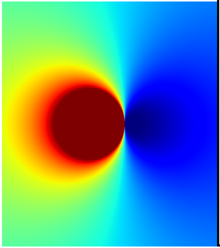

So life is simultaneously worse and better than discussed.

Is S' compact?

How about 3D? $(-x/|x|^3)$

Would like an analytical tool that requires 'less' fanciness.

Cauchy Principal Value

But I don't **want** to integrate across a singularity!

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$\rightarrow$ punch it out.

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$$\int_{-1}^{1} \frac{1}{x} dx \text{ not defined?}$$

$$\int_{-1}^{1} \frac{1}{x} dx := \lim_{\varepsilon \to 0+} \left(\int_{-1}^{-\varepsilon} \frac{1}{x} + \int_{\varepsilon}^{1} \frac{1}{x} \right)$$

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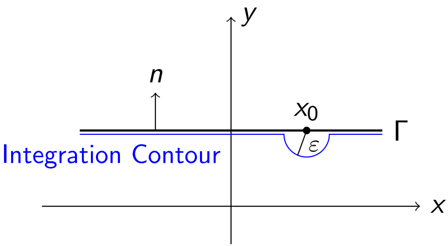
PV

Important: Symmetry matters!

$$\int_{-1}^{-2\varepsilon} \frac{1}{x} + \int_{\varepsilon}^{1} \frac{1}{x}$$

nD Principal Value



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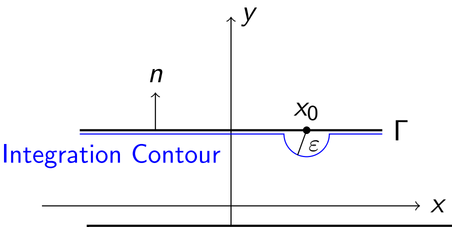
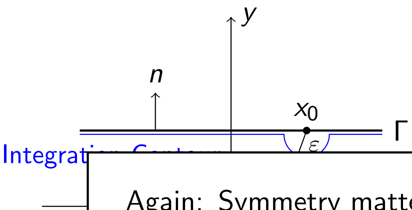


Again: Symmetry matters!

Not an ellipse,
not a potato,
a circle.



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Sphere in 3D.

nD Principal Value

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Even worse singularities:
“Hadamard finite part”

*n*D Principal Value

Integrat

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Sphere in 3D.

Even worse singularities:
"Hadamard finite part"

CPV integrals: "singular"

HFP integrals: "*hyper*singular"

Recap: Layer Potentials

$$(S\sigma)(x) := \int_\Gamma G(x-y)\sigma(y)ds_y$$

$$(S'\sigma)(x) := PV\ \hat{n}\cdot\nabla_x \int_\Gamma G(x-y)\sigma(y)ds_y$$

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$$(D'\sigma)(x) := f \ldots \int$$

Important for us:

Recover 'average' of interior and exterior limit without having to refer to off-surface values.

Green's Theorem, Green's Formula

Theorem (Green's Theorem [Kress LIE Thm 6.3])

$$\int_D u \triangle v + \nabla u \cdot \nabla v = \int_{\partial D} u(\hat{n} \cdot \nabla v) ds \qquad \text{(First)}$$

$$\int_D u \triangle v - v \triangle u = \int_{\partial D} u(\hat{n} \cdot \nabla v) - v(\hat{n} \cdot \nabla u) ds \qquad \text{(Second)}$$

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If $\triangle v = 0$, then

$$\int_{\partial D} \hat{n}\cdot\nabla v = ?$$

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What if $\triangle v = 0$ and $u = G(|y - x|)$ in Green's second identity?

Can you write that more briefly?

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Theorem (Green's Formula [Kress LIE Thm 6.5])

If $\triangle u = 0$*, then*

$$(S(\hat{n} \cdot \nabla u) - Du)(x) = \begin{cases} u(x) & x \in D \\ \frac{u(x)}{2} & x \in \partial D \\ 0 & x \notin D \end{cases}$$

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If $\triangle u = 0$, *then*

$$(S(\hat{n}\cdot\nabla u \quad \begin{cases} u(x) & x \in D \end{cases}$$

Know 'Cauchy data' of u
$\to$ compute u anywhere

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Theorem (Green's Formula [Kress LIE Thm 6.5])

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Normal derivative of Green's Formula?

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Normal derivative of Green's Formula?

What about Green's formula on the exterior part of D?

Things harmonic functions (don't) do

Theorem (Mean Value Theorem [Kress LIE Thm 6.7])

$$u(x) = \fint_{B(x,r)} u(y)dy = \fint_{\partial B(x,r)} u(y)dy$$

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Theorem (Mean Value Theorem [Kress LIE Thm 6.7])

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Define $\fint$?

Things harmonic functions (don't) do

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Define $\fint$?

Trace back to Green's Formula.

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If $\triangle u = 0$ on compact set $\bar{D}$:
u attains its maximum on the boundary.

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Theorem (Maximum Principle [Kress LIE 6.9])

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Suppose it were to attain its maximum somewhere inside an open set. . .

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Theorem (Maximum Principle [Kress LIE 6.9])

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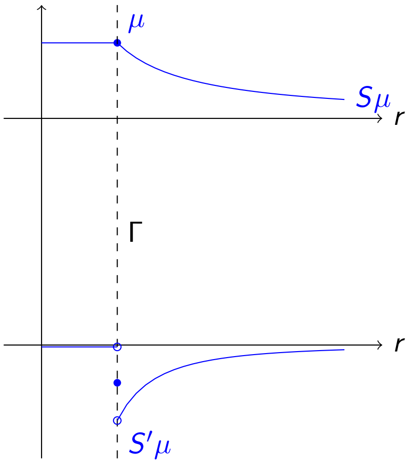
So boundaries are special.

What do our *constructed* harmonic functions do there?

Jump relations: Intuition



Questions?

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