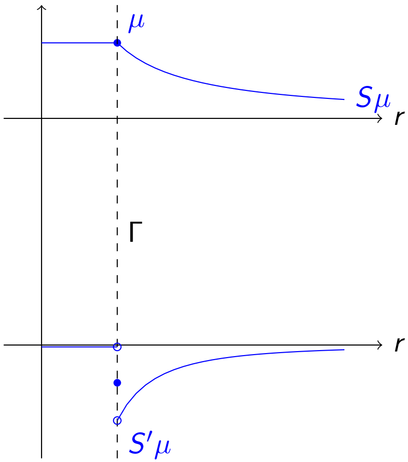
Integral Equations and Fast Algorithms
Lecture 12: Potential Theory

CS 598AK · October 3, 2013

Outline

Harmonic Potential Theory

Jump relations: Intuition



Jump relations for S

Theorem (Single layer jump relations [Kress LIE Thm. 6.14])

For $\partial\Omega$ twice continuously differentiable and $\varphi \in C(\partial\Omega)$:

$S\varphi(x)$ is continuous throughout $\mathbb{R}^n$.

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For $\partial\Omega$ twice continuously differentiable and $\varphi \in C(\partial\Omega)$:

$S\varphi(x)$ is continuous throughout $\mathbb{R}^n$.

Proof idea: Use cut-off function, uniform limit of continuous functions.

Jump relations for D

Theorem (Double layer jump relations [Kress LIE Thm. 6.17])

For $\partial\Omega$ twice continuously differentiable and $\varphi \in C(\partial\Omega)$:

- *$D\varphi(x)$ can be continuously extended from the exterior of Ω $\to D^+\varphi(x)$ for $x \in \partial\Omega$*
- *$D\varphi(x)$ can be continuously extended from the interior of $\Omega \to$ $D^-\varphi(x)$ for $x \in \partial\Omega$*

$$D^\pm\varphi(x) = D\varphi(x) \pm \frac{\varphi}{2}$$

$D\varphi(x)$ exists everywhere as an (at worst) improper integral.

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Consider density 1. Use $\int n \cdot \nabla G = 0$.

Jump relations for S'

Theorem (S' jump relations [Kress LIE Thm. 6.18])

For $\partial\Omega$ twice continuously differentiable and $\varphi \in C(\partial\Omega)$:

- *$S'\varphi(x)$ can be continuously extended from the exterior of Ω $\to (S')^+\varphi(x)$ for $x \in \partial\Omega$*
- *$S'\varphi(x)$ can be continuously extended from the interior of Ω $\to (S')^-\varphi(x)$ for $x \in \partial\Omega$*

$$(S')^{\pm}\varphi(x) = S'\varphi(x) \mp \frac{\varphi}{2}$$

$S'\varphi(x)$ exists everywhere as an (at worst) improper integral.

Jump relations: overview

Let $[X] = X_+ - X_-$. (Normal points towards "+" = "exterior".)

$$[S\sigma] = 0$$

$$\lim_{x \to x_0 \pm} (S'\sigma) = \left(S' \mp \frac{1}{2} I\right)(\sigma)(x_0) \qquad \Rightarrow \qquad [S'\sigma] = -\sigma$$

$$\lim_{x \to x_0 \pm} (D\sigma) = \left(D \pm \frac{1}{2} I\right)(\sigma)(x_0) \qquad \Rightarrow \qquad [D\sigma] = \sigma$$

$$[D'\sigma] = 0$$

See [1] and [2] for jumps of "non-standard" layer potentials.

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$$[D'\sigma] = 0$$

See [1] and [2] for jumps of "non-standard" layer potentials.

Spot the bonus fact.

Green's formula and infinity

$\Omega \subseteq \mathbb{R}^n$ bounded, C^1, connected boundary, $\triangle u = 0$, u bounded

$$(S_{\partial\Omega}(\hat{n} \cdot \nabla u) - D_{\partial\Omega} u)(x) + (S_{\partial B_r}(\hat{n} \cdot \nabla u) - D_{\partial B_r} u)(x) = u(x)$$

for x between $\partial\Omega$ and B_r.

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Now $r \to \infty$.

Behavior of individual terms?

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Behavior of individual terms?

Theorem (Green's Formula in the exterior [Kress LIE Thm 6.10])

$$(S_{\partial\Omega}(\hat{n} \cdot \nabla u) - D_{\partial\Omega} u)(x) + u_\infty = u(x)$$

for some constant u_∞. Only for $n = 2$,

$$u_\infty = \frac{1}{2\pi r} \int_{|y|=r} u(y) ds_y.$$

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Every bounded harmonic function is representable as...

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How about its derivatives?

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How about its derivatives?

$$u(x) = u_\infty + O\left(\frac{1}{|x|}\right)$$
$$\nabla u(x) = O\left(\frac{1}{|x|^{n-1}}\right)$$

Questions?

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