Integral Equations and Fast Algorithms
Lecture 13: Potential Theory, BVPs

CS 598AK · October 8, 2013

HW3 bits

- Kernel interface/handling the discontinuity
- Counting iterations

Outline

Harmonic Potential Theory

Recap: Fredholm Alternative

Theorem (Fredholm Alternative [Kress LIE Thm. 4.14])

$A : X \to X$ compact.

Then either:

- $I - A$ *and* $I - A^*$ *are bijective*

or:

- $\dim N(I - A) = \dim N(I - A^*)$
- $(I - A)(X) = N(I - A^*)^\perp$
- $(I - A^*)(X) = N(I - A)^\perp$

Jump relations: overview

Let $[X] = X_+ - X_-$. (Normal points towards "+"="exterior".)

$$[S\sigma] = 0$$

$$\lim_{x \to x_0 \pm} (S'\sigma) = \left(S' \mp \frac{1}{2} I \right) (\sigma)(x_0) \qquad \Rightarrow \qquad [S'\sigma] = -\sigma$$

$$\lim_{x \to x_0 \pm} (D\sigma) = \left(D \pm \frac{1}{2} I \right) (\sigma)(x_0) \qquad \Rightarrow \qquad [D\sigma] = \sigma$$

$$[D'\sigma] = 0$$

See [1] and [2] for jumps of "non-standard" layer potentials.

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Spot the bonus fact.

Boundary Value Problems, Uniqueness

Find $u \in C(\bar{\Omega})$ with $\triangle u = 0$ such that

	Dirichlet	**Neumann**
Int.	$\lim_{x \to \partial\Omega-} u(x) = g$	$\lim_{x \to \partial\Omega-} \hat{n} \cdot \nabla u(x) = g$
Ext.	$\lim_{x \to \partial\Omega+} u(x) = g$	$\lim_{x \to \partial\Omega+} \hat{n} \cdot \nabla u(x) = g$

with $g \in C(\partial\Omega)$.

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Find IE representations for each.

Questions?

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