Integral Equations and Fast Algorithms
Lecture 14: BVPs continued

CS 598AK · October 10, 2013

Outline

Harmonic Potential Theory

Recap: Fredholm Alternative

Theorem (Fredholm Alternative [Kress LIE Thm. 4.14])

$A : X \to X$ *compact.*

Then either:

- $I - A$ *and* $I - A^*$ *are bijective*

or:

- $\dim N(I - A) = \dim N(I - A^*)$
- $(I - A)(X) = N(I - A^*)^\perp$
- $(I - A^*)(X) = N(I - A)^\perp$

Jump relations: overview

Let $[X] = X_+ - X_-$. (Normal points towards "+"="exterior".)

$$[S\sigma] = 0$$

$$\lim_{x \to x_0 \pm} (S'\sigma) = \left(S' \mp \frac{1}{2} I \right) (\sigma)(x_0) \qquad \Rightarrow \qquad [S'\sigma] = -\sigma$$

$$\lim_{x \to x_0 \pm} (D\sigma) = \left(D \pm \frac{1}{2} I \right) (\sigma)(x_0) \qquad \Rightarrow \qquad [D\sigma] = \sigma$$

$$[D'\sigma] = 0$$

See [1] and [2] for jumps of "non-standard" layer potentials.

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See [1] and [2] for jumps of "non-standard" layer potentials.

Spot the bonus fact.

Boundary Value Problems, Uniqueness

Find $u \in C(\bar{\Omega})$ with $\triangle u = 0$ such that

	Dirichlet	**Neumann**
Int.	$\lim_{x \to \partial\Omega -} u(x) = g$	$\lim_{x \to \partial\Omega -} \hat{n} \cdot \nabla u(x) = g$
Ext.	$\lim_{x \to \partial\Omega +} u(x) = g$	$\lim_{x \to \partial\Omega +} \hat{n} \cdot \nabla u(x) = g$

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Ext.	$\lim_{x \to \partial\Omega+} u(x) = g$ $u(x) = \begin{cases} O(1) & 2D \\ o(1) & 3D \end{cases}$ as $\lvert x \rvert \to \infty$	$\lim_{x \to \partial\Omega+} \hat{n} \cdot \nabla u(x) = g$ $u(x) = o(1)$ as $\lvert x \rvert \to \infty$

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What's a DtN map?

Find IE representations for each.

Uniqueness of IE solutions?

Theorem (Nullspaces [Kress LIE Thm 6.20])

- $N(I/2 - D) = N(I/2 - S') = \{0\}$
- $N(I/2 + D) = \text{span}\{1\}$, $N(I/2 + S') = \text{span}\{\psi\}$, *where* $\int \psi \neq 0$.

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$\to$ "Clean" Existence for 3 out of 4.

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Problem: $N(I/2 + S') = \{\psi\}\ldots$ but we do not know ψ.

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Note: Singularity only at origin! (assumed $\in \Omega$)

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- Existence/uniqueness?

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$\rightarrow$ Existence for 4 out of 4.

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> $\to$ Existence for 4 out of 4.
>
> Remaining key shortcoming of IE theory for BVPs?

Questions?

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