Integral Equations and Fast Algorithms
Lecture 14: BVPs continued

CS 598AK · October 10, 2013
Outline

Harmonic Potential Theory
Recap: Fredholm Alternative

**Theorem (Fredholm Alternative [Kress LIE Thm. 4.14])**

Let \( A : X \to X \) be compact.

Then either:

1. \( I - A \) and \( I - A^* \) are bijective

or:

2. \( \dim N(I - A) = \dim N(I - A^*) \)
3. \( (I - A)(X) = N(I - A^*)^\perp \)
4. \( (I - A^*)(X) = N(I - A)^\perp \)
Jump relations: overview

Let \([X] = X_+ - X_-\). (Normal points towards “+” = “exterior”.)

\[
\lim_{x \to x_0 \pm} (S'\sigma) = \left(S' \mp \frac{1}{2} I\right)(\sigma)(x_0) \Rightarrow [S'\sigma] = -\sigma
\]

\[
\lim_{x \to x_0 \pm} (D\sigma) = \left(D \pm \frac{1}{2} I\right)(\sigma)(x_0) \Rightarrow [D\sigma] = \sigma
\]

\[
[S\sigma] = 0
\]

\[
[D'\sigma] = 0
\]

Let \([X] = X_+ - X_-.\) (Normal points towards “+” = “exterior”.)

\[
\lim_{x \to x_0^\pm} (S'\sigma) = \left( S' \mp \frac{1}{2} I \right)(\sigma)(x_0) \quad \Rightarrow \quad [S'\sigma] = -\sigma
\]

\[
\lim_{x \to x_0^\pm} (D\sigma) = \left( D \pm \frac{1}{2} I \right)(\sigma)(x_0) \quad \Rightarrow \quad [D\sigma] = \sigma
\]

\[
[S\sigma] = 0
\]

\[
[D'\sigma] = 0
\]


Spot the bonus fact.
**Boundary Value Problems, Uniqueness**

Find \( u \in C(\bar{\Omega}) \) with \( \triangle u = 0 \) such that

<table>
<thead>
<tr>
<th></th>
<th><strong>Dirichlet</strong></th>
<th><strong>Neumann</strong></th>
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<tbody>
<tr>
<td><strong>Int.</strong></td>
<td>( \lim_{x \to \partial \Omega_-} u(x) = g )</td>
<td>( \lim_{x \to \partial \Omega_-} \hat{n} \cdot \nabla u(x) = g )</td>
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with \( g \in C(\partial \Omega) \).
Find $u \in C(\bar{\Omega})$ with $\triangle u = 0$ such that

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<td>$u(x) = \begin{cases} O(1) &amp; 2D \ o(1) &amp; 3D \end{cases}$ as $</td>
<td>x</td>
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with $g \in C(\partial \Omega)$. 

What does $f(x) = O(1)$ mean? (and $f(x) = o(1)$?)
Find $u \in C(\bar{\Omega})$ with $\triangle u = 0$ such that

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with $g \in C(\partial \Omega)$. 

**Laplace**
Find \( u \in C(\bar{\Omega}) \) with \( \triangle u = 0 \) such that

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with \( g \in C(\partial \Omega) \).

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Find $u \in C(\bar{\Omega})$ with $\triangle u = 0$ such that

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<tr>
<td>unique</td>
<td>may differ by constant</td>
</tr>
<tr>
<td><strong>Ext.</strong></td>
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with $g \in C(\partial \Omega)$.

What does $f(x) = O(1)$ mean? (and $f(x) = o(1)$?)

Dirichlet uniqueness: why? (Hint: Maximum principle)
Find $u \in C(\bar{\Omega})$ with $\triangle u = 0$ such that

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with $g \in C(\partial \Omega)$.

What does $f(x) = O(1)$ mean? (and $f(x) = o(1)$?)

Dirichlet uniqueness: why? (Hint: Maximum principle)

Neumann uniqueness: why? (Hint: Green's first theorem)

\[ \int_{\Omega} u \triangle v + \nabla u \cdot \nabla v = \int_{\partial \Omega} u(\hat{n} \cdot \nabla v) ds \]
Boundary Value Problems, Uniqueness

Find \( u \in C(\bar{\Omega}) \) with \( \triangle u = 0 \) such that

\[
\text{Dirichlet}\quad \lim_{x \to \partial \Omega^-} u(x) = g
\]

\[\downarrow \text{unique}\]

\[
\text{Ext.}\quad \lim_{x \to \partial \Omega^+} u(x) = h
\]

\[\downarrow \text{unique}\]

with \( g \in C(\partial \Omega) \).

What does \( f(x) = O(1) \) mean? (and \( f(x) = o(1) \)?)

Dirichlet uniqueness: why? (Hint: Maximum principle)

Neumann uniqueness: why? (Hint: Green’s first theorem)

\[
\int_{\Omega} u \nabla v + \nabla u \cdot \nabla v = \int_{\partial \Omega} u(\hat{n} \cdot \nabla v) ds
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Missing assumptions on \( \Omega \)?
Find $u \in C(\bar{\Omega})$ with $\Delta u = 0$ such that

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Dirichlet uniqueness: why?  
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Neumann uniqueness: why?  
(Hint: Green’s first theorem)

$$\int_{\Omega} u \Delta v + \nabla u \cdot \nabla v = \int_{\partial \Omega} u(\hat{n} \cdot \nabla v) \, ds$$

Missing assumptions on $\Omega$?

What’s a DtN map?
Find \( u \in C(\bar{\Omega}) \) with \( \Delta u = 0 \) such that

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\[ \int_{\Omega} u \Delta v + \nabla u \cdot \nabla v = \int_{\partial \Omega} u(\hat{n} \cdot \nabla v) ds \]

Missing assumptions on \( \Omega \)?

What's a DtN map?

Find IE representations for each.
Uniqueness of IE solutions?

Theorem (Nullspaces [Kress LIE Thm 6.20])

- $N(I/2 - D) = N(I/2 - S') = \{0\}$
- $N(I/2 + D) = \text{span}\{1\}$, $N(I/2 + S') = \text{span}\{\psi\}$, where $\int \psi \neq 0$. 
Uniqueness of IE solutions?

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What to show?

Start with \( I/2 - D \). What BVP?

Hint: Use jump relations, use BVP uniqueness, use both kinds of boundary data.

Next: \( I/2 + D \) on exterior. Show \( \phi \) constant. Nullspace identified?

Extra conditions on RHS?

\( (I - A)(X) = N(I - A^*) \perp \) Laplace
Uniqueness of IE solutions?

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Extra conditions on RHS?

\[
(I - A)(X) = N(I - A^*)^\perp
\]

→ “Clean” Existence for 3 out of 4.
Patching up Exterior Dirichlet Problem: $N(I/2 + S') = \{\psi\} \ldots$ but we do not know $\psi$. 

Use a different kernel: 

$\hat{n} \cdot \nabla y G(x, y) \rightarrow \hat{n} \cdot \nabla y G(x, y) + 1 |x|^{n-2}$ 

Note: Singularity only at origin! (assumed $\in \Omega$) 

• 2D behavior? 3D behavior? 
• Still a solution of the PDE? 
• Compact? 
• Jump condition? Exterior limit? Deduce $u(x) = 0$ on exterior. 
• $|x|^{n-2} u(x) = ?$ on exterior 
• Thus $\hat{\psi} = 0$. Contribution of the second term? 
• $\psi/2 + \partial \psi = 0$, i.e. $\psi \in N(I/2 + D) = ?$ 
• Existence/uniqueness? 

$\rightarrow$ Existence for 4 out of 4.
Patching up Exterior Dirichlet Problem: \[ N(I/2 + S') = \{ \psi \} \ldots \text{but we do not know } \psi. \]

Use a different kernel:

\[ \hat{n} \cdot \nabla_y G(x, y) \rightarrow \hat{n} \cdot \nabla_y G(x, y) + \frac{1}{|x|^{n-2}} \]

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- \( |x|^{n-2} u(x) = ? \) on exterior
Patching up Exterior Dirichlet Problem: \( N(I/2 + S') = \{\psi\} \ldots \) but we do not know \( \psi \).

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- Thus \( \int \varphi = 0 \). Contribution of the second term?
Patching up Exterior Dirichlet Problem: $N(I/2 + S^{'}) = \{\psi\} \ldots$ but we do not know $\psi$.

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- Thus $\int \varphi = 0$. Contribution of the second term?
- $\varphi/2 + D\varphi = 0$, i.e. $\varphi \in N(I/2 + D) = ?$
Patching up Exterior Dirichlet Problem: \( N(I/2 + S') = \{\psi\} \ldots \) but we do not know \( \psi \).

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- Existence/uniqueness?
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- Existence/uniqueness?

\[ \rightarrow \text{Existence for 4 out of 4.} \]
Patching up Exterior Dirichlet Problem: \( N(I/2 + S') = \{\psi\} \ldots \) but we do not know \( \psi \).

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- Still a solution of the PDE?
- Compact?
- Jump condition? Exterior limit? Deduce \( u = 0 \) on exterior.
- \( |x|^{n-2} u(x) = ? \) on exterior
- Thus \( \int \varphi = 0 \). Consider \( \varphi/2 + D\varphi = 0 \), i.e.
- Existence/uniqueness?

\[ \rightarrow \] Existence for 4 out of 4.

Remaining key shortcoming of IE theory for BVPs?