Integral Equations and Fast Algorithms
Lecture 15: Helmholtz

CS 598AK · October 15, 2013

Outline

Harmonic Potential Theory

Uniqueness of IE solutions?

Theorem (Nullspaces [Kress LIE Thm 6.20])

- $N(I/2 - D) = N(I/2 - S') = \{0\}$
- $N(I/2 + D) = \text{span}\{1\}$, $N(I/2 + S') = \text{span}\{\psi\}$, *where* $\int \psi \neq 0$.

Uniqueness of IE solutions?

Theorem (Nullspaces [Kress LIE Thm 6.20])

- $N(I/2 - D) = N(I/2 - S') = \{0\}$
- $N(I/2 + D) = \text{span}\{1\}$, $N(I/2 + S') = \text{span}\{\psi\}$, *where* $\int \psi \neq 0$.

What to show?

Uniqueness of IE solutions?

Theorem (Nullspaces [Kress LIE Thm 6.20])

- $N(I/2 - D) = N(I/2 - S') = \{0\}$
- $N(I/2 + D) = \text{span}\{1\}$, $N(I/2 + S') = \text{span}\{\psi\}$, *where* $\int \psi \neq 0$.

What to show?

Start with $I/2 - D$. What BVP?

Uniqueness of IE solutions?

Theorem (Nullspaces [Kress LIE Thm 6.20]**)**

- $N(I/2 - D) = N(I/2 - S') = \{0\}$
- $N(I/2 + D) = \text{span}\{1\}$, $N(I/2 + S') = \text{span}\{\psi\}$, *where* $\int \psi \neq 0$.

What to show?

Start with $I/2 - D$. What BVP?

Hint: Use jump relations, use BVP uniqueness, use both kinds of boundary data.

Uniqueness of IE solutions?

Theorem (Nullspaces [Kress LIE Thm 6.20])

- $N(I/2 - D) = N(I/2 - S') = \{0\}$
- $N(I/2 + D) = \text{span}\{1\}$, $N(I/2 + S') = \text{span}\{\psi\}$, *where* $\int \psi \neq 0$.

What to show?

Start with $I/2 - D$. What BVP?

Hint: Use jump relations, use BVP uniqueness, use both kinds of boundary data.

Next: $I/2 + D$ on exterior.

Uniqueness of IE solutions?

Theorem (Nullspaces [Kress LIE Thm 6.20])

- $N(I/2 - D) = N(I/2 - S') = \{0\}$
- $N(I/2 + D) = \operatorname{span}\{1\}$, $N(I/2 + S') = \operatorname{span}\{\psi\}$, *where* $\int \psi \neq 0$.

What to show?

Start with $I/2 - D$. What BVP?

Hint: Use jump relations, use BVP uniqueness, use both kinds of boundary data.

Next: $I/2 + D$ on exterior.

Show φ constant. Nullspace identified?

Uniqueness of IE solutions?

Theorem (Nullspaces [Kress LIE Thm 6.20]**)**

- $N(I/2 - D) = N(I/2 - S') = \{0\}$
- $N(I/2 + D) = \text{span}\{1\}$, $N(I/2 + S') = \text{span}\{\psi\}$, *where* $\int \psi \neq 0$.

What to show?

Start with $I/2 - D$. What BVP?

Hint: Use jump relations, use BVP uniqueness, use both kinds of boundary data.

Next: $I/2 + D$ on exte

Show φ constant. Nul

Extra conditions on RHS?
$(I - A)(X) = N(I - A^*)^{\perp}$

Uniqueness of IE solutions?

Theorem (Nullspaces [Kress LIE Thm 6.20]**)**

- $N(I/2 - D) = N(I/2 - S') = \{0\}$
- $N(I/2 + D) = \text{span}\{1\}$, $N(I/2 + S') = \text{span}\{\psi\}$, *where* $\int \psi \neq 0$.

What to show?

Start with $I/2 - D$. What BVP?

Hint: Use jump relations, use BVP uniqueness, use both kinds of boundary data.

Next: $I/2 + D$ on exte

Show φ constant. Nul

Extra conditions on RHS?
$(I - A)(X) = N(I - A^*)^\perp$

$\rightarrow$ "Clean" Existence for 3 out of 4.

Patching up Exterior Dirichlet

Problem: $N(I/2 + S') = \{\psi\}\ldots$ but we do not know ψ.

Patching up Exterior Dirichlet

Problem: $N(I/2 + S') = \{\psi\}\ldots$ but we do not know ψ.

Use a different kernel:

$$\hat{n} \cdot \nabla_y G(x, y) \qquad \to \qquad \hat{n} \cdot \nabla_y G(x, y) + \frac{1}{|x|^{n-2}}$$

Note: Singularity only at origin! (assumed $\in \Omega$)

Patching up Exterior Dirichlet

Problem: $N(I/2 + S') = \{\psi\}$... but we do not know ψ.

Use a different kernel:

$$\hat{n} \cdot \nabla_y G(x, y) \qquad \to \qquad \hat{n} \cdot \nabla_y G(x, y) + \frac{1}{|x|^{n-2}}$$

Note: Singularity only at origin! (assumed $\in \Omega$)

- 2D behavior? 3D behavior?

Patching up Exterior Dirichlet

Problem: $N(I/2 + S') = \{\psi\}\ldots$ but we do not know ψ.

Use a different kernel:

$$\hat{n} \cdot \nabla_y G(x, y) \qquad \to \qquad \hat{n} \cdot \nabla_y G(x, y) + \frac{1}{|x|^{n-2}}$$

Note: Singularity only at origin! (assumed $\in \Omega$)

- 2D behavior? 3D behavior?
- Still a solution of the PDE?

Patching up Exterior Dirichlet

Problem: $N(I/2 + S') = \{\psi\} \ldots$ but we do not know ψ.

Use a different kernel:

$$\hat{n} \cdot \nabla_y G(x, y) \qquad \to \qquad \hat{n} \cdot \nabla_y G(x, y) + \frac{1}{|x|^{n-2}}$$

Note: Singularity only at origin! (assumed $\in \Omega$)

- 2D behavior? 3D behavior?
- Still a solution of the PDE?
- Compact?

Patching up Exterior Dirichlet

Problem: $N(I/2 + S') = \{\psi\}\ldots$ but we do not know ψ.

Use a different kernel:

$$\hat{n} \cdot \nabla_y G(x, y) \qquad \rightarrow \qquad \hat{n} \cdot \nabla_y G(x, y) + \frac{1}{|x|^{n-2}}$$

Note: Singularity only at origin! (assumed $\in \Omega$)

- 2D behavior? 3D behavior?
- Still a solution of the PDE?
- Compact?
- Jump condition? Exterior limit? Deduce $u = 0$ on exterior.

Patching up Exterior Dirichlet

Problem: $N(I/2 + S') = \{\psi\}\dots$ but we do not know ψ.

Use a different kernel:

$$\hat{n} \cdot \nabla_y G(x, y) \qquad \rightarrow \qquad \hat{n} \cdot \nabla_y G(x, y) + \frac{1}{|x|^{n-2}}$$

Note: Singularity only at origin! (assumed $\in \Omega$)

- 2D behavior? 3D behavior?
- Still a solution of the PDE?
- Compact?
- Jump condition? Exterior limit? Deduce $u = 0$ on exterior.
- $|x|^{n-2} u(x) = ?$ on exterior

Patching up Exterior Dirichlet

Problem: $N(I/2 + S') = \{\psi\}$... but we do not know ψ.

Use a different kernel:

$$\hat{n} \cdot \nabla_y G(x, y) \qquad \to \qquad \hat{n} \cdot \nabla_y G(x, y) + \frac{1}{|x|^{n-2}}$$

Note: Singularity only at origin! (assumed $\in \Omega$)

- 2D behavior? 3D behavior?
- Still a solution of the PDE?
- Compact?
- Jump condition? Exterior limit? Deduce $u = 0$ on exterior.
- $|x|^{n-2} u(x) = ?$ on exterior
- Thus $\int \varphi = 0$. Contribution of the second term?

Patching up Exterior Dirichlet

Problem: $N(I/2 + S') = \{\psi\}$... but we do not know ψ.

Use a different kernel:

$$\hat{n} \cdot \nabla_y G(x, y) \qquad \rightarrow \qquad \hat{n} \cdot \nabla_y G(x, y) + \frac{1}{|x|^{n-2}}$$

Note: Singularity only at origin! (assumed $\in \Omega$)

- 2D behavior? 3D behavior?
- Still a solution of the PDE?
- Compact?
- Jump condition? Exterior limit? Deduce $u = 0$ on exterior.
- $|x|^{n-2} u(x) = ?$ on exterior
- Thus $\int \varphi = 0$. Contribution of the second term?
- $\varphi/2 + D\varphi = 0$, i.e. $\varphi \in N(I/2 + D) = ?$

Patching up Exterior Dirichlet

Problem: $N(I/2 + S') = \{\psi\}\ldots$ but we do not know ψ.

Use a different kernel:

$$\hat{n} \cdot \nabla_y G(x, y) \qquad \rightarrow \qquad \hat{n} \cdot \nabla_y G(x, y) + \frac{1}{|x|^{n-2}}$$

Note: Singularity only at origin! (assumed $\in \Omega$)

- 2D behavior? 3D behavior?
- Still a solution of the PDE?
- Compact?
- Jump condition? Exterior limit? Deduce $u = 0$ on exterior.
- $|x|^{n-2} u(x) = ?$ on exterior
- Thus $\int \varphi = 0$. Contribution of the second term?
- $\varphi/2 + D\varphi = 0$, i.e. $\varphi \in N(I/2 + D) = ?$
- Existence/uniqueness?

Patching up Exterior Dirichlet

Problem: $N(I/2 + S') = \{\psi\}\ldots$ but we do not know ψ.

Use a different kernel:

$$\hat{n} \cdot \nabla_y G(x,y) \qquad \to \qquad \hat{n} \cdot \nabla_y G(x,y) + \frac{1}{|x|^{n-2}}$$

Note: Singularity only at origin! (assumed $\in \Omega$)

- 2D behavior? 3D behavior?
- Still a solution of the PDE?
- Compact?
- Jump condition? Exterior limit? Deduce $u = 0$ on exterior.
- $|x|^{n-2} u(x) = ?$ on exterior
- Thus $\int \varphi = 0$. Contribution of the second term?
- $\varphi/2 + D\varphi = 0$, i.
- Existence/uniquer

$\to$ Existence for 4 out of 4.

Patching up Exterior Dirichlet

Problem: $N(I/2 + S') = \{\psi\}$... but we do not know ψ.

Use a different kernel:

$$\hat{n} \cdot \nabla_y G(x, y) \qquad \rightarrow \qquad \hat{n} \cdot \nabla_y G(x, y) + \frac{1}{|x|^{n-2}}$$

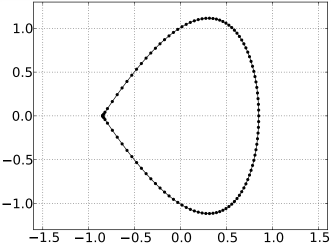
Note: Singularity only at origin! (assumed $\in \Omega$)

- 2D behavior? 3D behavior?
- Still a solution of the PDE?
- Compact?
- Jump condition? Exterior limit? Deduce $u = 0$ on exterior.
- $|x|^{n-2}u(x) = ?$ or
- Thus $\int \varphi = 0$. Co
- $\varphi/2 + D\varphi = 0$, i.
- Existence/uniquer

$\rightarrow$ Existence for 4 out of 4.

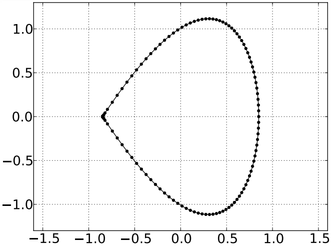
Remaining key shortcoming of IE theory for BVPs?

Domains with corners



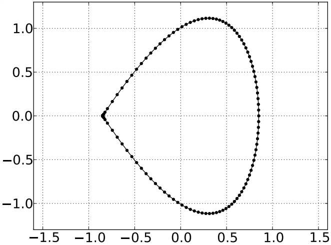
What's the problem?

Domains with corners



What's the problem? *(Hint: Jump condition for constant density)*

Domains with corners

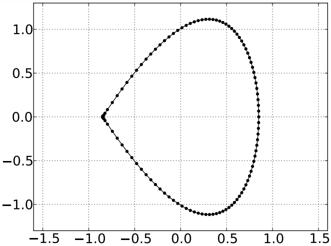


What's the problem? *(Hint: Jump condition for constant density)*

At corner x_0: (2D)

$$\lim_{x \to x_0 \pm} = \int_{\partial\Omega} \hat{n} \cdot \nabla_y G(x,y) \varphi(y) ds_y \pm \frac{1}{2} \frac{\langle \text{opening angle on } \pm \text{ side} \rangle}{\pi} \varphi$$

Domains with corners



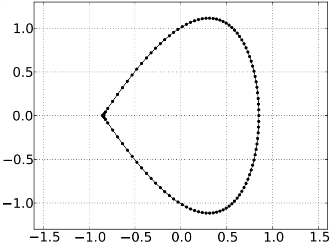
What's the problem? *(Hint: Jump condition for constant density)*

At corner x_0: (2D)

$$\lim_{x \to x_0 \pm} = \int_{\partial\Omega} \hat{n} \cdot \nabla_y G(x,y)\varphi(y) ds_y \pm \frac{1}{2} \frac{\langle \text{opening angle on } \pm \text{ side} \rangle}{\pi} \varphi$$

$\rightarrow$ non-continuous behavior of potential on Γ at x_0
What space have we been living in?

Domains with corners



What's the problem? *(Hint: Jump condition for constant density)*

At corner x_0: (2D)

$$\lim_{x \to x_0 \pm} = \int_{\partial\Omega} \hat{n} \cdot \nabla_y G(x$$

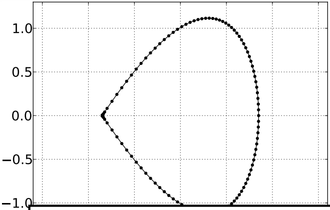
$\rightarrow$ non-continuous beh

What space have we b

Fixes:

- I + Bounded (Neumann) + Compact (Fredholm)
- Use L^2 theory (point behavior "invisible")

Domains with corners



What's the problem?

At corner x_0: (2D)

$$\lim_{x \to x_0 \pm} = \int_{\partial\Omega} \hat{n} \cdot \nabla_y G(x$$

→ non-continuous beh

What space have we b

Fixes:

- I + Bounded (Neumann) + Compact (Fredholm)
- Use L^2 theory (point behavior "invisible")

Numerically: Needs consideration, but ultimately easy to fix.

Questions?

?