Integral Equations and Fast Algorithms
Lecture 16: Helmholtz

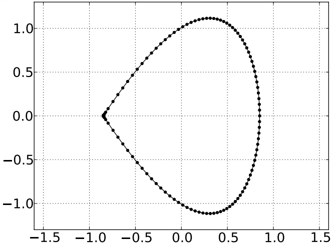
CS 598AK · October 17, 2013

Outline

Harmonic Potential Theory

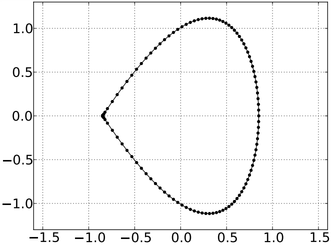
Helmholtz Potential Theory

Domains with corners



What's the problem?

Domains with corners



What's the problem? *(Hint: Jump condition for constant density)*

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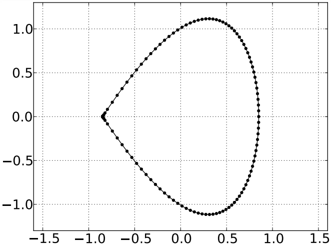

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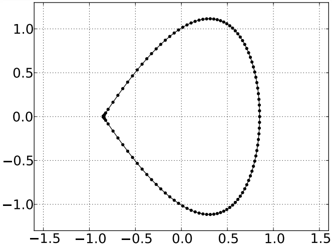
At corner x_0: (2D)

$$\lim_{x\to x_0\pm} = PV \int_{\partial\Omega} \hat{n}\cdot\nabla_y G(x,y)\varphi(y) ds_y \pm \frac{1}{2}\frac{\langle\text{opening angle on} \pm \text{side}\rangle}{\pi}\varphi$$

Domains with corners



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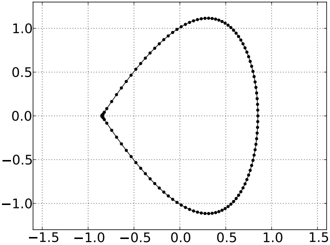
→ non-continuous behavior of potential on Γ at x_0

What space have we been living in?

Domains with corners



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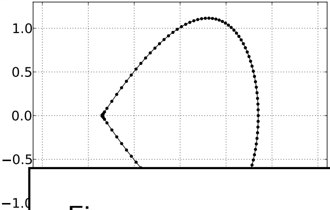

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Numerically: Needs consideration, but ultimately straightforward to fix. (once you know how)

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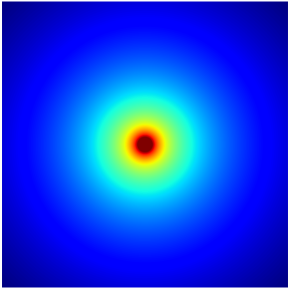
Helmholtz Potential Theory

Helmholtz

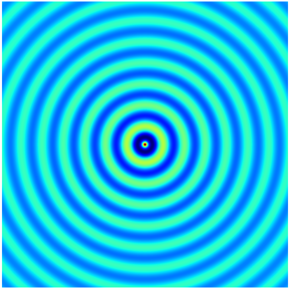
Laplace Equation

$-\triangle u = \delta$



Helmholtz Equation

$\triangle u + k^2 u = \delta$



(where $k = \omega / c$)

Helmholtz

Laplace Equation

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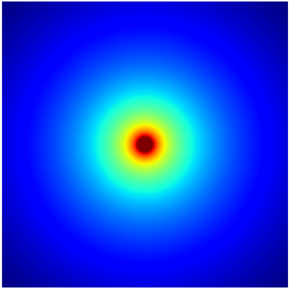

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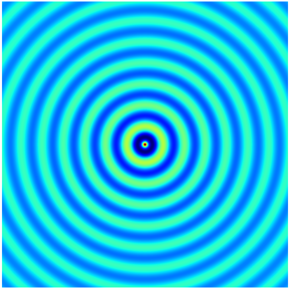

What happens for increasing k?

Helmholtz

Laplace Equation

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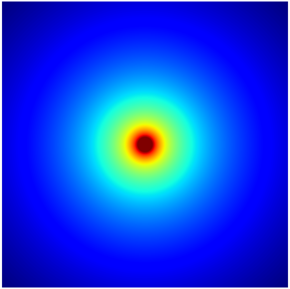

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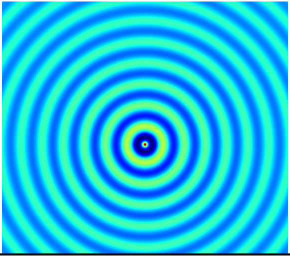

What happens for increasing k?

...and complex k?

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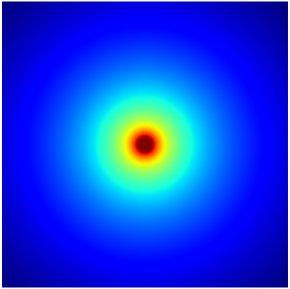

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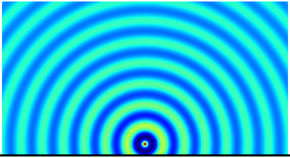

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What is c and where does it come from?

Helmholtz

Laplace Equation

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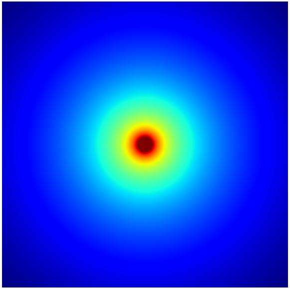

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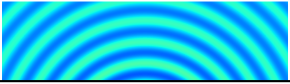

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$\partial_t^2 u = c^2 \triangle u$

Helmholtz

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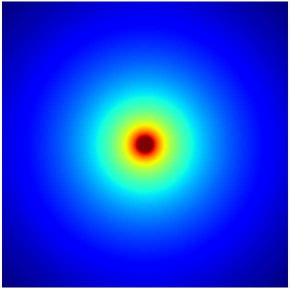

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How to recover the time-dependent solution?

Helmholtz

Laplace Equation

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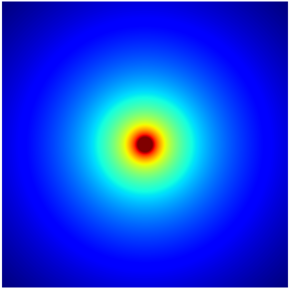

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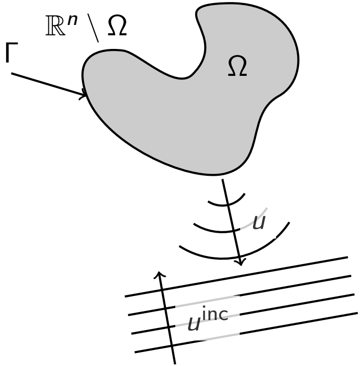
How to recover the time-dependent solution?

Typical problems being solved this way?

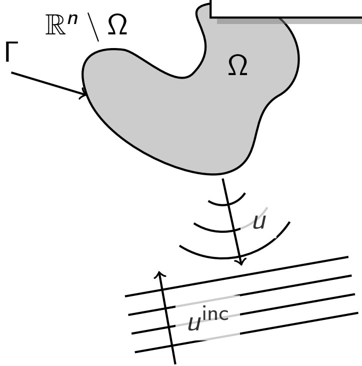
Scattering problem



$$u^{\text{tot}} = u + u^{\text{inc}}$$

Scattering problem

Where do the two couple?



$$u^{\text{tot}} = u + u^{\text{inc}}$$

Back to physics

Physical quantities:

- Velocity potential: $U(x,t) = u(x)e^{-i\omega t}$
- Velocity: $v = (1/\rho_0)\nabla U$
- Pressure: $p = -\partial_t U = i\omega u e^{-i\omega t}$
  - Equation of state: $p = f(\rho)$

(fix phase by e.g. taking real part)

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What's ρ_0?

What happens to a pressure BC as $\omega \to 0$?

Fundamental solutions

$$G_k(x) = \begin{cases} \frac{i}{4} H_0^{(1)}(k|x|) & \text{2D} \\ \frac{1}{4\pi} \frac{e^{ik|x|}}{|x|} & \text{3D} \end{cases}$$

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Hankel function?

Separation of variables in polar coordinates $\rightarrow$ Bessel ODE:

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - \alpha^2) y = 0$$

$H_\alpha^{(1)} = J_\alpha + iY_\alpha,$
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Asymptotic behavior for small k?

$G_k(x)/G_0(x) \to ?$ $(x \to 0)$

What happens for $-k$?

Helmholtz Boundary Conditions

- **Sound-soft**: Pressure remains constant
  - Scatterer "gives"

  $u = f \to$ Dirichlet
- **Sound-hard**: Pressure same on both sides of interface
  - Scatterer "does not give"

  $\hat{n} \cdot \nabla u = 0 \to$ Neumann
- **Impedance**: Some pressure translates into motion
  - Scatterer "resists"

  $\hat{n} \cdot \nabla u + ik\lambda u = 0 \to$ Robin $(\lambda > 0)$
- **Sommerfeld** radiation condition: allow only outgoing waves

$$(r \to \infty)$$

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$$r^{\frac{n-1}{2}} \left(\frac{\partial}{\partial r} - ik \right) u(x) \rightarrow 0 \qquad (r \rightarrow \infty)$$

Helmholtz Boundary Conditions

Many interesting BCs $\rightarrow$ many IEs! :)

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Helmholtz Boundary Conditions

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>
> Transmission between media: What's continuous?

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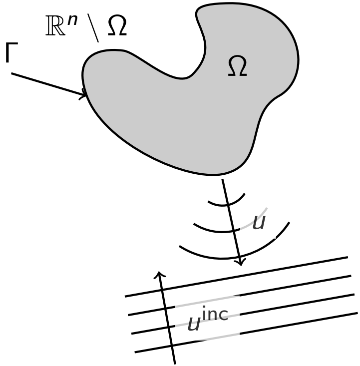

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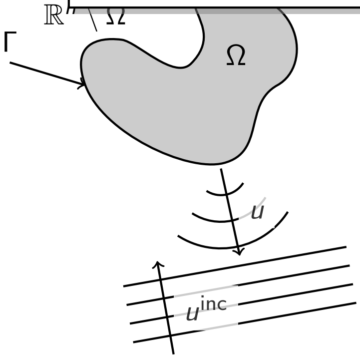
Scattering problem

What's allowed as an incoming wave?



$$u^{\text{tot}} = u + u^{\text{inc}}$$

Unchanged from Laplace

Theorem (Green's Formula [Colton/Kress IAEST Thm 2.1]**)**

If $\triangle u + k^2 u = 0$, *then*

$$(S(\hat{n} \cdot \nabla u) - Du)(x) = \begin{cases} u(x) & x \in D \\ \frac{u(x)}{2} & x \in \partial D \\ 0 & x \notin D \end{cases}$$

$$[Su] = 0$$

$$\lim_{x \to x_0 \pm} (S'u) = \left(S' \mp \frac{1}{2} I \right)(u)(x_0) \quad \Rightarrow \quad [S'u] = -u$$

$$\lim_{x \to x_0 \pm} (Du) = \left(D \pm \frac{1}{2} I \right)(u)(x_0) \quad \Rightarrow \quad [Du] = u$$

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Green's thm reminder:

$$\int_\Omega u \triangle v - v \triangle u = \int_{\partial\Omega} u(\hat{n}\cdot\nabla v) - v(\hat{n}\cdot\nabla u) ds$$

Resonances

$-\triangle$ on a bounded (interior) domain with homogeneous Dirichlet/Neumann BCs has countably many real, positive eigenvalues.

Problem?

Resonances

$-\triangle$ on a bounded (interior) domain with homogeneous Dirichlet/Neumann BCs has countably many real, positive eigenvalues.

Problem?

$$-\triangle u = \lambda u$$

$$\triangle u + k^2 u = 0$$

Questions?

?