Integral Equations and Fast Algorithms
Lecture 17: Helmholtz BVPs, IEs

CS 598AK · October 22, 2013

Outline

Helmholtz Potential Theory

Helmholtz Boundary Conditions

- **Sound-soft**: Pressure remains constant
  - Scatterer "gives"

  $u = f \to$ Dirichlet
- **Sound-hard**: Pressure same on both sides of interface
  - Scatterer "does not give"

  $\hat{n} \cdot \nabla u = 0 \to$ Neumann
- **Impedance**: Some pressure translates into motion
  - Scatterer "resists"

  $\hat{n} \cdot \nabla u + ik\lambda u = 0 \to$ Robin $(\lambda > 0)$
- **Sommerfeld** radiation condition: allow only outgoing waves

$$(r \to \infty)$$

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Many interesting BCs $\to$ many IEs! :)

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Transmission between media: What's continuous?

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Unchanged from Laplace

Theorem (Green's Formula [Colton/Kress IAEST Thm 2.1])

If $\triangle u + k^2 u = 0$, *then*

$$(S(\hat{n} \cdot \nabla u) - Du)(x) = \begin{cases} u(x) & x \in D \\ \frac{u(x)}{2} & x \in \partial D \\ 0 & x \notin D \end{cases}$$

$$[Su] = 0$$

$$\lim_{x \to x_0 \pm} (S' u) = \left(S' \mp \frac{1}{2} I \right) (u)(x_0) \quad \Rightarrow \quad [S' u] = -u$$

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Green's thm reminder:

$$\int_\Omega u \triangle v - v \triangle u = \int_{\partial\Omega} u(\hat{n} \cdot \nabla v) - v(\hat{n} \cdot \nabla u) ds$$

Asymptotics

Let $\triangle u + k^2 u = 0$ with Sommerfeld.

Theorem (Far Field Pattern [Colton/Kress IAEST Thm 2.6 and (3.85)]**)**

$$u(x) = \frac{e^{ik|x|}}{|x|^{(n-1)/2}} \left(u_\infty \left(\frac{x}{|x|} \right) + O\left(\frac{1}{|x|} \right) \right)$$

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Something missing?

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Theorem (Rellich's Lemma [Colton/Kress IAEST Lemma 2.12])

$$\|u\|_{L^2(\partial B(0,r))} \to 0 \quad as \quad r \to \infty \qquad \Rightarrow \qquad u = 0.$$

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Theorem (Rellich's Lemma [Colton/Kress IAEST Lemma 2.12])

$$\|u\|_{L^2(\partial B(0,r))}$$

First step towards uniqueness.
But: really need criterion on $\partial\Omega$ (not $\partial B(0, r)$)

Getting to Uniqueness

Let $\triangle u + k^2 u = 0$ with Sommerfeld.

Theorem (Ext. uniqueness helper [Colton/Kress IAEST Thm 2.13]**)**

If

$$\operatorname{Im} \int_{\partial\Omega} u \overline{(\hat{n} \cdot \nabla u)} ds \geq 0$$

then $u = 0$ *in* $\mathbb{R}^3 \setminus \bar{\Omega}$.

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- $\Omega_r := B(0,r) \setminus \Omega$
- Apply Green's theorem $\int_\Omega u \triangle v + \nabla u \cdot \nabla v = \int_{\partial\Omega} u(\hat{n} \cdot \nabla v) ds$ on Ω_r to $\int_{\partial\Omega_r} u\overline{(\hat{n} \cdot \nabla u)} ds$

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- Use imaginary part in first bit.
- Use Rellich.

Resonances

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Problem?

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Problem?

$$-\triangle u = \lambda u$$

$$\triangle u + k^2 u = 0$$

Boundary Value Problems, Uniqueness

Find $u \in C(\bar{D})$ with $\triangle u + k^2 = 0$ such that

	Dirichlet	**Neumann**
Int.	$\lim_{x \to \partial D-} u(x) = g$	$\lim_{x \to \partial D-} \hat{n} \cdot \nabla u(x) = g$
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Find IE representations for each.

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Issue: Ext. IE inherits non-uniqueness from 'adjoint' int. BVP

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Uniqueness for remaining IEs similar:

- Set RHS of IE to 0.
- Use uniqueness to get zero limit on one side.
- Use jump condition to get zero limit on other side.
- Go to "other" jump condition to get zero limit on other side.
- Use jump condition to show density = 0.

$\Rightarrow$ Existence for all four BVPs.

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Also valid for Laplace

Questions?

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