Integral Equations and Fast Algorithms
Lecture 18: Calderón, Intro numerics

CS 598AK · October 24, 2013

Outline

Calderón identities

- D' is self-adjoint
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Also valid for Laplace

</ 1 >

Theory

< 2 >

Numerics

Outline

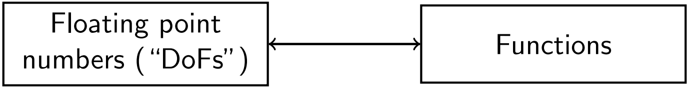
Helmholtz Potential Theory

Intro to Numerics

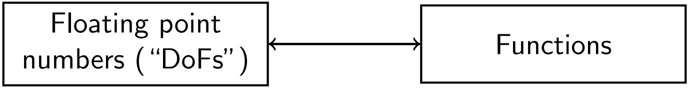
What do we need?

- Discretize curves and surfaces
  - Interpolation
  - Grid management
  - Adaptivity
- Discretize densities
- Discretize integral equations
  - Nyström, Collocation, Galerkin
- Compute integrals on them
  - “Smooth” quadrature
  - Singular quadrature
- Solve linear systems

Constructing discrete function spaces



Constructing discrete function spaces



Discretization relies on three things:

- Base/reference domain
- Basis of functions
- Meaning of DoFs

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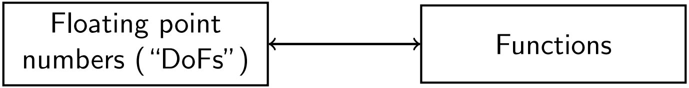

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> Related finite element concept:
> *Ciarlet triple*

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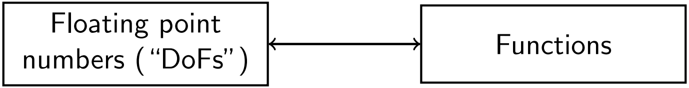

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Related finite element concept:
Ciarlet triple

Discretization options for a curve?

DoF choices

Common DoF choices:

- Point values of function
- Point values of (directional?) derivatives
- Basis coefficients
- Moments

Often: useful to have both "modes", "nodes", jump back and forth

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First order	Fifth order
1,000 DoFs $\approx$ 1,000 triangles Error: 0.1	1,000 DoFs $\approx$ 66 triangles Error: 0.1

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Why h

Order p: Error bounded as

$$\|u_h -$$

Want $p \geq 3$ available.

Assumption: Solution sufficiently smooth

Ideally: p chosen by user

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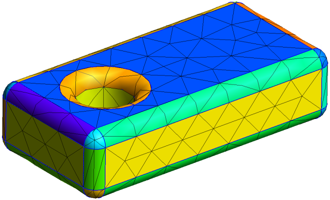

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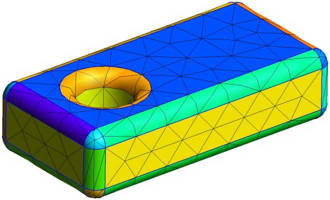

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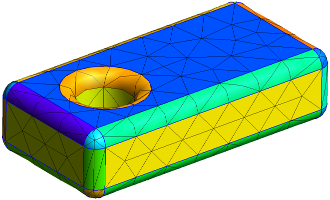
Why have an unstructured mesh?

- Adaptable to many engineering problems
- Deal with topology
- Deal with solution non-smoothness
- Adaptivity in space
- Adaptivity in time



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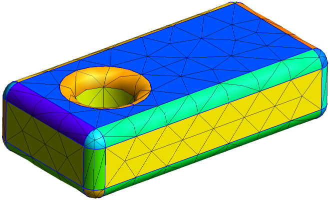

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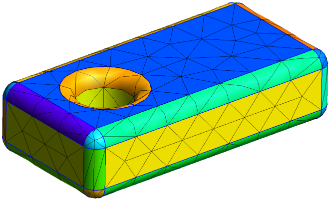

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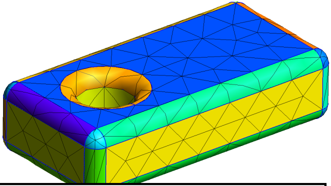

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CAD demo

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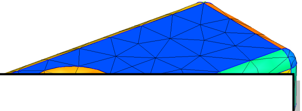

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What reference elements are used? Why?

Questions?

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