Integral Equations and Fast Algorithms
Lecture 19: High-order numerics

CS 598AK · October 29, 2013

Outline

Fixed-order vs spectral

Fixed-order	Spectral
Number of DoFs n	Number of DoFs n
$\sim$	$\sim$
Number of "discr. units"	Number of modes resolved
$$\text{Error} \sim \frac{1}{n^p}$$	$$\text{Error} \sim \frac{1}{C^n}$$

Fixed-order vs spectral

Fixed-order

Number of DoFs n

$\sim$

Number of "discr. units"

$$\text{Error} \sim \frac{1}{n^p}$$

Examples?

Spectral

Number of DoFs n

$\sim$

Number of modes resolved

$$\text{Error} \sim \frac{1}{C^n}$$

Fixed-order vs spectral

Fixed-order

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Examples?

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Examples?

Fixed-order vs spectral

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Examples?	Examples?

Assumptions buried in each?

Fixed-order vs spectral

Fixed-order

Number of DoFs n

$\sim$

Number of "discr. units"

$$\text{Error} \sim \frac{1}{n^p}$$

Examples?

Spectral

Number of DoFs n

$\sim$

Number of modes resolved

Assumptions buried in each?

Natural DoF match:

- Fixed-order: point values
- Spectral: modal coefficients

Fixed-order vs spectral

Fixed-order

Number of DoFs n

$\sim$

Number of "discr. units"

$$\text{Error} \sim \frac{1}{n^p}$$

Examples?

Spectral

Number of DoFs n

Assumptions buried in each?

Natural DoF match:

- Fixed-order: point values
- Spectral: modal coefficients

Difficulty with purely modal discretization?

Fixed-order vs spectral

Fixed-order

Number of DoFs n

$\sim$

Number of "discr. units"

$$\text{Error} \sim \frac{1}{n^p}$$

Examples?

Spectral

Number of DoFs n

Assumptions buried in each?

Natural DoF match:

- Fixed-order: point values
- Spectral: modal coefficients

Difficulty with purely modal discretization? "Pseudospectral"

Fixed-order vs spectral

Fixed-order	Spectral
Number of DoFs n $\sim$ Number of "discr. units"	

$$\text{Error} \sim \frac{1}{n^p}$$

Examples?

Assumptions buried in each?

Natural DoF match:

- Fixed-order: point values
- Spectral: modal coefficients

Difficulty with purely modal discretization? "Pseudospectral"

Spectral + unstructured = ?

Vandermonde matrices

$$\begin{pmatrix} x_0^0 & x_0^1 & \cdots & x_0^n \\ x_1^0 & x_1^1 & \cdots & x_1^n \\ \vdots & \vdots & \ddots & \vdots \\ x_n^0 & x_n^1 & \cdots & x_n^n \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = ?$$

Generalized Vandermonde matrices

$$\begin{pmatrix} \varphi_0(x_0) & \varphi_1(x_0) & \cdots & \varphi_n(x_0) \\ \varphi_0(x_1) & \varphi_1(x_1) & \cdots & \varphi_n(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_0(x_n) & \varphi_1(x_n) & \cdots & \varphi_n(x_n) \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = ?$$

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Instantly generalizes to multiple dimensions

Generalized Vandermonde matrices

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Instantly generalizes to multiple dimensions

Invertible?

Generalized Vandermonde matrices

$$\begin{pmatrix} \varphi_0(x_0) & \varphi_1(x_0) & \cdots & \varphi_n(x_0) \\ \varphi_0(x_1) & \varphi_1(x_1) & \cdots & \varphi_n(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_0(x_n) & \varphi_1(\ldots & & \end{pmatrix} \begin{pmatrix} \text{DAL COEFFS} \end{pmatrix} = \begin{pmatrix} \text{DAL COEFFS} \end{pmatrix}$$

Instantly generalizes to multiple dimensions

Invertible?

Conditioning?

Interpolation node placement demo

Vandermonde conditioning demo

2D basis demo

Questions?

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