Admin bits

• Homework?
• Office hours?
• Books
• Video? Hope whiteboard is readable
Today

Integral Equations: what? (cont’d)

Integral equations: why?

Spaces

Operators
Outline

Integral Equations: what? (cont’d)

Integral equations: why?

Spaces

Operators
Solving a BVP with integral equations

Solve a (interior Laplace Dirichlet) BVP, \( \partial \Omega = \Gamma \)

\[
\triangle u = 0 \quad \text{in} \Omega, \quad u|_{\Gamma} = f|_{\Gamma}.
\]

1. Pick representation:

\[
 u(x) := (S\sigma)(x)
\]

2. Take (interior) limit onto \( \Gamma \):

\[
 u|_{\Gamma} = S\sigma
\]

3. Enforce BC:

\[
 u|_{\Gamma} = f
\]

4. Solve resulting linear system:

\[
 S\sigma = f
\]

5. Obtain PDE solution in \( \Omega \) by evaluating representation
What to do?

1. Pick representation:
   \[ u(x) := (D\sigma)(x) \]

2. Take (interior) limit onto \( \Gamma \):
   \[ u|_{\Gamma} = D\sigma - \sigma/2 \]

3. Enforce BC:
   \[ u|_{\Gamma} = f \]

4. Solve resulting linear system:
   \[ (D - \text{Id}/2)\sigma = f \]

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Lies & Damned lies

A few nontrivial issues swept under the rug:

- **Theory**
  Does this even have a right to work?

- **Singular Quadrature**
  Must compute (3D)

\[ S\sigma(x) = \int_{\Gamma} \frac{1}{|x - y|} \sigma(y) ds_y \]

at and near \(|x - y| = 0\) on \(\Gamma\)

- **Computational cost**
  - What is really going on?
  - What is the computational cost?
Lies & Damned lies

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- **Computational cost**
  - What is really going on?
  - What is the computational cost?

We’ll revisit these issues, in order.
Integral Equations: what? (cont’d)

Integral equations: why?

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Operators
Disclaimer: Silver bullets?

No known method is a ‘silver bullet’
(that matches or beats all competitors in every setting)

Most important goals of this class:
• intuitive understanding of where integral equations are well-suited
• overview of machinery involved
Disclaimer: Silver bullets?

No known method is a ‘silver bullet’
(that matches or beats all competitors in every setting)

Most important goals of this class:
• intuitive understanding of where integral

With that in mind:
Let’s dig up some dirt on the competition :)
Recap: Condition number

**Condition number**

Amplification factor of relative error in solving $Ax = b$

Assume $\tilde{b} = b + \Delta b$ — actually solved: $A\tilde{x} = \tilde{b}$.

\[
\frac{\text{rel. err in } \tilde{x}}{\text{rel. err in } \tilde{b}} = \frac{\|A^{-1}\Delta b\|/\|A^{-1}b\|}{\|\Delta b\|/\|b\|} = \frac{\|A^{-1}\Delta b\|}{\|\Delta b\|} \cdot \frac{\|b\|}{\|A^{-1}b\|} = \frac{\|A^{-1}\Delta b\|}{\|\Delta b\|} \cdot \frac{\|A(A^{-1}b)\|}{\|A^{-1}b\|} \leq \|A^{-1}\| \|A\| =: \kappa(A)
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Recap: Condition number

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\[
\kappa(A) = \sigma_{\max}(A)/\sigma_{\min}(A)
\]
Recap: Catastrophic cancellation

Floating point numbers:  \text{mantissa} \cdot 2^{\text{exponent}}

- 0.1101101101 (exponent: -2)
- 0.0000000000000001101101 (exponent: -8)
- 0.00001101101 (exponent: -4)
- 1.101101 (exponent: 0)

Impossible: Compute a very small number by subtracting very big numbers

What? Why? Spaces Operators
Recap: Catastrophic cancellation

Floating point numbers: \[ \text{mantissa} \cdot 2^{\text{exponent}} \]

Impossible:
Compute a very small number by subtracting very big numbers
Finite difference accuracy

Single precision FD error on $\sin(20 \cdot 2\pi)$

And that's just applying the forward operator.

How about inverting it?
Finite difference accuracy

Single precision FD error on $\sin(20 \cdot 2\pi)$

And that’s just applying the forward operator.
How about inverting it?
**Discretizing derivatives**

Derivative operator: Conditioning?

\[ \partial e^{\alpha x} = \alpha e^{\alpha x} \]

*Unbounded*, gets worse with better approximation

I.e. \[ \| \partial \| \to \infty \]

Practically usually: \[ \| \partial_h \| \sim \frac{1}{h} \]

\[ \partial (\text{const}) = 0 \]

Zero eigenvalue \(\to\) not invertible

I.e. \[ \| \partial^{-1} \| \ldots ? \]

Practically fixed (to some extent) by boundary conditions (but also unbounded below!)

In the limit \( h \to 0 \), the condition number does not exist.
Discretizing derivatives

Derivative operator: Conditioning?

\[ \partial e^{\alpha x} = \alpha e^{\alpha x} \quad \text{Unbounded, gets worse with better approximation} \]

\[ \partial (\text{const}) = 0 \]

Zero eigenvalue \(\Rightarrow\) not invertible

I.e. \(\|\partial\| \to \infty\)

I.e. \(\|\partial^{-1}\| \ldots ?\)

Practically usually: \(\|\partial_h\| \sim \frac{1}{h}\)

Practically fixed (to some extent) by boundary conditions (but also

In the limit \(h \to 0\), the

Problem in left column also addressed by multigrid.

But: not in this course :)

What? Why? Spaces Operators
“Morally correct” numerical schemes

A method is “morally correct” if it...

- can get machine precision
  - “What’s so special about 4/8/x digits?”
- has condition number $O(1)$ in problem size
  - i.e. “the method is scalable”
    - (≠ e.g. parallel scalability)
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;)}
Summary

Why IEs?

• $O(n)$ solve complexity (or, at worst, $O(n \log n)$)
• DOFs only on boundary
• Conditioning
• Exterior problems
• Geometric flexibility
  • e.g. moving boundaries—no volume mesh to deal with
• ‘Data-driven’

But:

• More machinery to learn, code
  • Many more choices of machinery
  • “Lots of flexibility”
• Less mature than FD/FE
• Favors (piecewise) constant-coefficient elliptic
  • Other PDEs/BVPs possible, but require work
Outline

Integral Equations: what? (cont’d)

Integral equations: why?

Spaces

Operators
Disclaimer

Some of what follows is *slightly* sloppy.

Use a book or Wikipedia to get the full truth.
A *norm* $\| \cdot \|$ maps an element of a *vector space* into $[0, \infty)$. It satisfies:

- $\|x\| = 0 \iff x = 0$
- $\|\lambda x\| = |\lambda| \|x\|$
- $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)
Norm

**Definition (Norm)**

A *norm* $\| \cdot \|$ maps an element of a *vector space* into $[0, \infty)$. It satisfies:

1. $\|x\| = 0 \iff x = 0$
2. $\|\lambda x\| = |\lambda| \|x\|$
3. $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)

Can create norm from *inner product*: $\|x\| = \sqrt{\langle x, x \rangle}$
Definition ($\ell^p$ norms)

$$
\|x\|_{\ell^p} := \sqrt[p]{\sum_{i=1}^{n} |x_n|^p}
$$

Most important: $p = 1, 2, \infty$. 
Finite-dimensional norms

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All equivalent. (Definition?)

\[ p = \infty! ? \]
**Definition (Convergent sequence)**

\[ x_n \to x \iff \|x_n - x\| \to 0 \]  
“convergence in norm”

**Definition (Cauchy sequence)**

For all \( \varepsilon > 0 \) there exists an \( n \) for which
\[ \|x_\nu - x_\mu\| \leq \varepsilon \text{ for } \mu, \nu \geq n \]
(Convergence without known limit)

**Definition (Complete/“Banach” space)**

Cauchy \( \Rightarrow \) Convergent
Convergence

Definition (Convergent sequence)
\[ x_n \to x :\iff \|x_n - x\| \to 0 \quad \text{“convergence in norm”} \]

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Definition (Complete/“Banach” space)
Cauchy \( \Rightarrow \) Convergent

Complete space with inner product: \textit{Hilbert space}
Convergence

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**Definition (Complete/“Banach” space)**

Cauchy \( \Rightarrow \) Convergent

Complete space with inner product: **Hilbert space**

Counterexample?
## Function spaces

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<thead>
<tr>
<th>Space</th>
<th>Description</th>
<th>Definition</th>
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<td>f(x)</td>
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</tr>
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$\Omega \subseteq \mathbb{R}^n$
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Many spaces exist in “local” flavors. Definition?
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Many spaces exist in “local” flavors. Definition?

Again $p = 1, 2, \infty$ usually.
Outline

Integral Equations: what? (cont’d)

Integral equations: why?

Spaces

Operators
Linear operators

$X, Y$: Banach spaces

$A : X \rightarrow Y$ linear operator
Linear operators

$X, Y$: Banach spaces

$A : X \rightarrow Y$ linear operator

What does linear mean here?
Linear operators

$X, Y$: Banach spaces

$A : X \to Y$ linear operator
Linear operators

$X, Y$: Banach spaces

$A : X \rightarrow Y$ linear operator

**Definition (Operator norm)**

$\|A\| := \sup \{ \|Ax\|/\|x\| : 0 \neq x \in X \}$
Linear operators

$X, Y$: Banach spaces

$A : X \rightarrow Y$ linear operator

**Definition (Operator norm)**

$$
\|A\| := \sup\{\|Ax\|/\|x\| : 0 \neq x \in X\}
$$

**Theorem**

$$
\|A\| \text{ bounded } \iff A \text{ continuous}
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$X, Y$: Banach spaces

$A : X \rightarrow Y$ linear operator

**Definition (Operator norm)**

$\|A\| := \sup\{\|Ax\|/\|x\| : 0 \neq x \in X\}$

**Theorem**

$\|A\|$ bounded $\iff A$ continuous

Is there a notion of ‘continuous at $x$’ for linear operators?
Linear operators

$X, Y$: Banach spaces

$A : X \to Y$ linear operator

**Definition (Operator norm)**

$$\|A\| := \sup\{\|Ax\|/\|x\| : 0 \neq x \in X\}$$

**Theorem**

$\|A\| \text{ bounded } \iff A \text{ continuous}$

Is there a notion of ‘continuous at $x$’ for linear operators?

Come up with a convenient test for boundedness.