Integral Equations and Fast Algorithms
Lecture 2: Functional Analysis/Intro Int.Eq.

CS 598AK · August 29, 2013

Admin bits

- Homework?
- Office hours?
- Books
- Video? Hope whiteboard is readable

Today

Integral Equations: what? (cont'd)

Integral equations: why?

Spaces

Operators

Outline

Solving a BVP with integral equations

Solve a (interior Laplace Dirichlet) BVP, $\partial\Omega = \Gamma$

$$\triangle u = 0 \quad \text{in}\Omega, \qquad u|_\Gamma = f|_\Gamma.$$

1. Pick representation:

$$u(x) := (S\sigma)(x)$$

2. Take (interior) limit onto Γ:

$$u|_\Gamma = S\sigma$$

3. Enforce BC:

$$u|_\Gamma = f$$

4. Solve resulting linear system:

$$S\sigma = f$$

5. Obtain PDE solution in Ω by evaluating representation

What to do?

1. Pick representation:

$$u(x) := (D\sigma)(x)$$

2. Take (interior) limit onto Γ:

$$u|_\Gamma = D\sigma - \sigma/2$$

3. Enforce BC:

$$u|_\Gamma = f$$

4. Solve resulting linear system:

$$(D - \operatorname{Id}/2)\sigma = f$$

5. Obtain PDE solution in Ω by evaluating representation

Wha[t to do?]

'Second-kind' integral equation
Previous one: 'First-kind'

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Lies & Damned lies

A few nontrivial issues swept under the rug:

- Theory
  Does this even have a right to work?
- Singular Quadrature
  Must compute (3D)

  $$S\sigma(x) = \int_\Gamma \frac{1}{|x-y|}\sigma(y)ds_y$$

  at and near $|x-y| = 0$ on Γ
- Computational cost
  - What is really going on?
  - What is the computational cost?

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We'll revisit these issues, in order.

Outline

Disclaimer: Silver bullets?

No known method is a 'silver bullet'

(that matches or beats all competitors in every setting)

Most important goals of this class:

- intuitive understanding of where integral equations are well-suited
- overview of machinery involved

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- intuitive understanding of where integral

With that in mind:
Let's dig up some dirt on the competition :)

Recap: Condition number

Condition number

Amplification factor of relative error in solving $Ax = b$

Assume $\tilde{b} = b + \Delta b$ — actually solved: $A\tilde{x} = \tilde{b}$.

$$\begin{aligned}\frac{\text{rel. err in } \tilde{x}}{\text{rel. err in } \tilde{b}} &= \frac{\|A^{-1}\Delta b\|/\|A^{-1}b\|}{\|\Delta b\|/\|b\|} = \frac{\|A^{-1}\Delta b\|}{\|\Delta b\|} \cdot \frac{\|b\|}{\|A^{-1}b\|} \\ &= \frac{\|A^{-1}\Delta b\|}{\|\Delta b\|} \cdot \frac{\|A(A^{-1}b)\|}{\|A^{-1}b\|} \\ &\leq \|A^{-1}\| \, \|A\| \\ &=: \kappa(A)\end{aligned}$$

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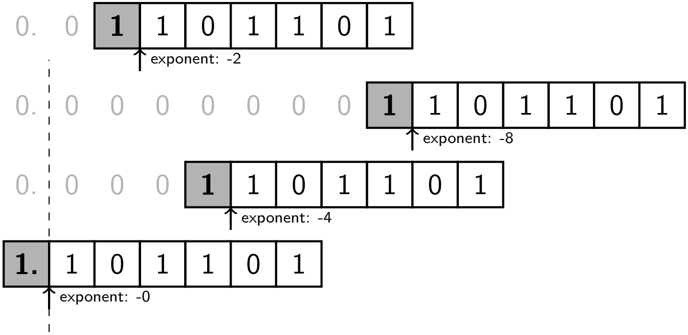
$$\kappa(A) = \sigma_{\max}(A) / \sigma_{\min}(A)$$

Recap: Catastrophic cancellation

Floating point numbers: mantissa $\cdot\, 2^{\text{exponent}}$



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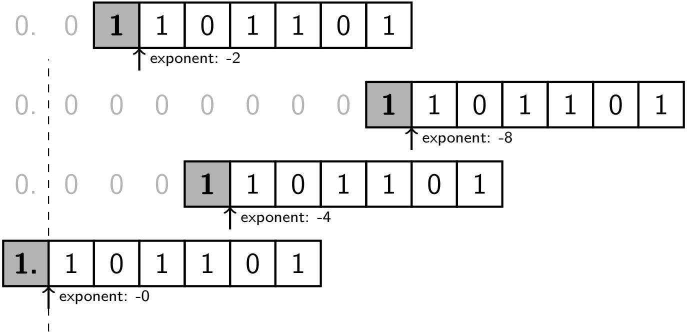
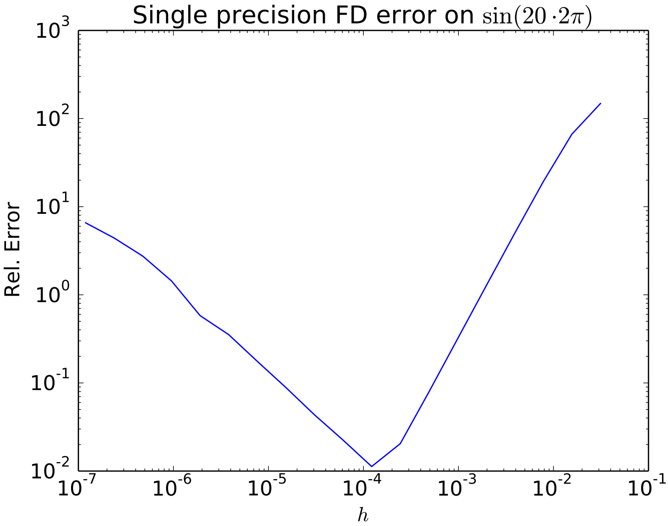


Impossible:
Compute a very small number by subtracting very big numbers

Finite difference accuracy



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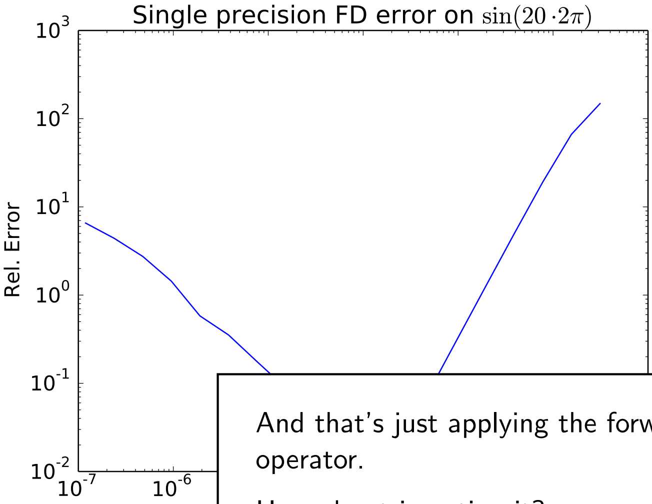

And that's just applying the forward operator.

How about inverting it?

Discretizing derivatives

Derivative operator: Conditioning?

$\partial e^{\alpha x} = \alpha e^{\alpha x}$ *Unbounded*, gets worse with better approximation	$\partial(\text{const}) = 0$ Zero eigenvalue $\to$ not invertible
I.e. $\|\partial\| \to \infty$	I.e. $\|\partial^{-1}\| \ldots ?$
Practically usually: $\|\partial_h\| \sim \frac{1}{h}$	Practically fixed (to some extent) by boundary conditions (but also unbounded below!)

In the limit $h \to 0$, the condition number does not exist.

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Problem in left column also addressed by multigrid.
But: not in this course :)

"Morally correct" numerical schemes

A method is "*morally correct*" if it...

- can get machine precision
  - "What's so special about $4/8/x$ digits?"
- has condition number $O(1)$ in problem size
  - i.e. "the method is scalable"
    ($\neq$ e.g. parallel scalability)

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Summary

Why IEs?

- $O(n)$ solve complexity (or, at worst, $O(n \log n)$)
- DOFs only on boundary
- Conditioning
- Exterior problems
- Geometric flexibility
  - e.g. moving boundaries–no volume mesh to deal with
- 'Data-driven'

But:

- More machinery to learn, code
  - Many more *choices* of machinery
  - "Lots of flexibility"
- Less mature than FD/FE
- Favors (piecewise) constant-coefficient elliptic
  - Other PDEs/BVPs possible, but require work

Outline

Disclaimer

Some of what follows is *slightly* sloppy.

Use a book or Wikipedia to get the full truth.

Norm

Definition (Norm)

A *norm* $\|\cdot\|$ maps an element of a *vector space* into $[0, \infty)$. It satisfies:

- $\|x\| = 0 \Leftrightarrow x = 0$
- $\|\lambda x\| = |\lambda| \|x\|$
- $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)

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Can create norm from *inner product*: $\|x\| = \sqrt{\langle x, x \rangle}$

Finite-dimensional norms

Definition (ℓ^p norms)

$$\|x\|_{\ell^p} := \sqrt[p]{\sum_{i=1}^{n} |x_n|^p}$$

Most important: $p = 1, 2, \infty$.

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$p = \infty$!?

Convergence

Definition (Convergent sequence)

$x_n \to x :\Leftrightarrow \|x_n - x\| \to 0$ "convergence in norm"

Definition (Cauchy sequence)

For all $\varepsilon > 0$ there exists an n for which
$\|x_\nu - x_\mu\| \leq \varepsilon$ for $\mu, \nu \geq n$
(Convergence without known limit)

Definition (Complete/"Banach" space)

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Counterexample?

Function spaces

$C(\Omega)$	f continuous, $\|f\|_\infty := \sup_{x\in\Omega} |f(x)|$
$C^k(\Omega)$	f k-times continuously differentiable
$C^{0,\alpha}(\Omega)$	$\|f\|_\alpha := \|f\|_\infty + \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$ $(\alpha \in (0,1))$
$C_L(\Omega)$	$|f(x) - (y)| \leq L\|x-y\|$
$L^p(\Omega)$	$\|f\|_p := \sqrt[p]{\int_D |f(x)|^p dx} < \infty$

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Again $p = 1, 2, \infty$ usually.

Outline

Linear operators

X, Y: Banach spaces

$A : X \to Y$ linear operator

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What does linear mean here?

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Is there a notion of 'continuous at x' for linear operators?

Come up with a convenient test for boundedness.

Questions?

?