Integral Equations and Fast Algorithms
Lecture 21: IE discretizations

CS 598AK · November 5, 2013

Patching up interior Neumann

Interior Neumann is non-unique as a BVP, so. . . ?

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Theorem (Nullspaces [Kress LIE Thm 6.20])

- $N(I/2 + D) = \text{span}\{1\}$,
- $N(I/2 + S') = \text{span}\{\psi\}$, *where* $\int \psi \neq 0$.

Representation: $u(x) = S\sigma(x)$
Integral equation: $S'\sigma + \sigma/2 =$ Neumann data g

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Want: $\int \sigma = 0 \quad \Rightarrow \quad \sigma \notin N(I/2 + S')$.

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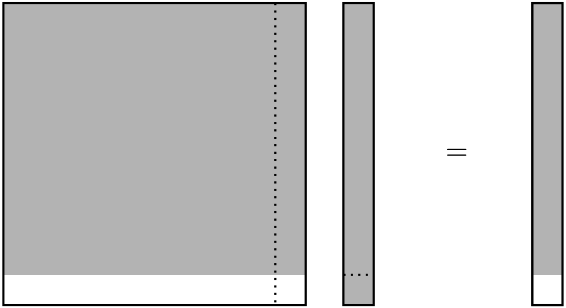

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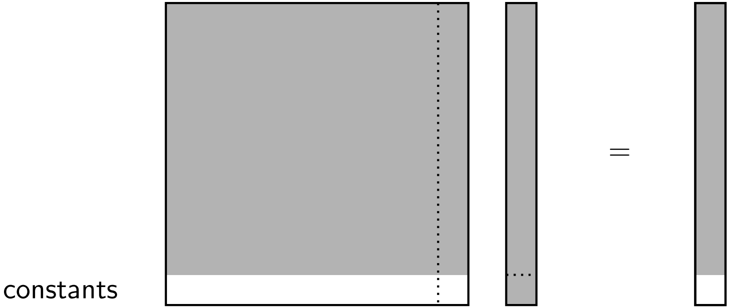

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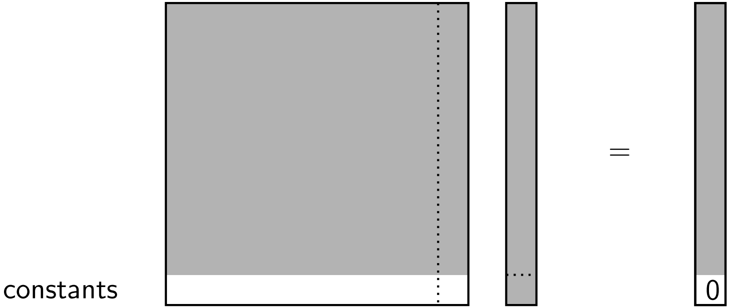

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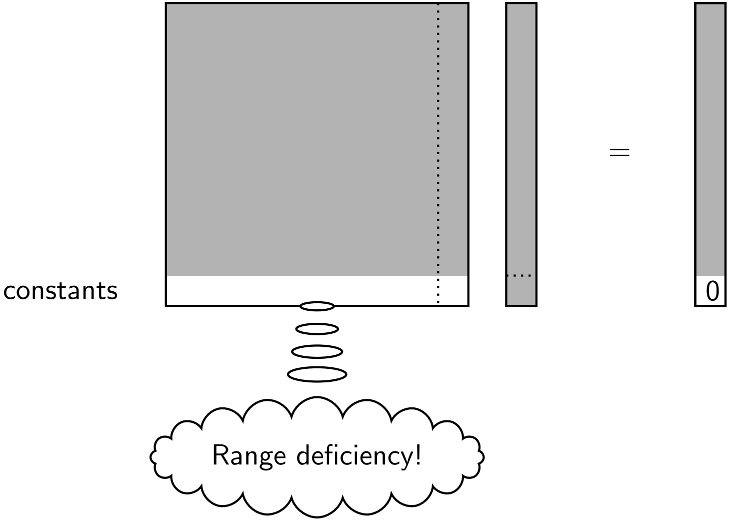

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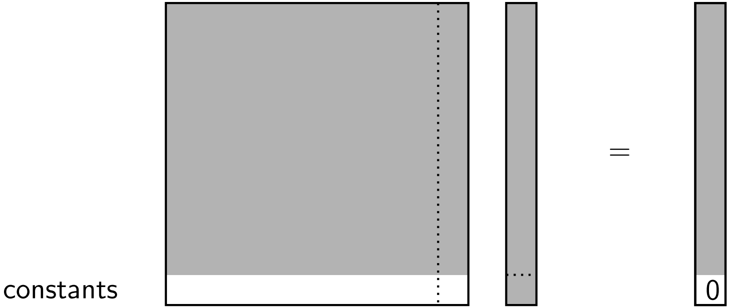

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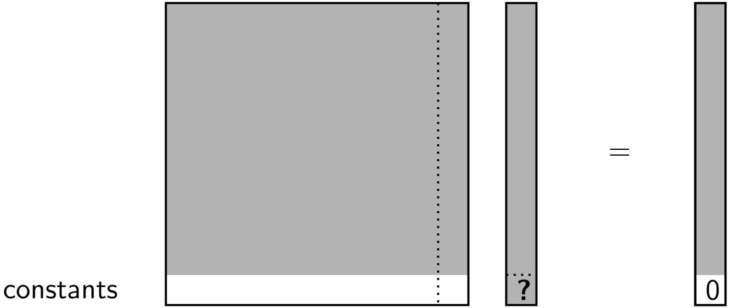

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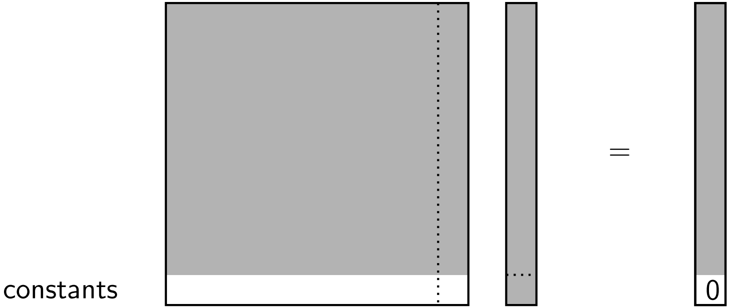

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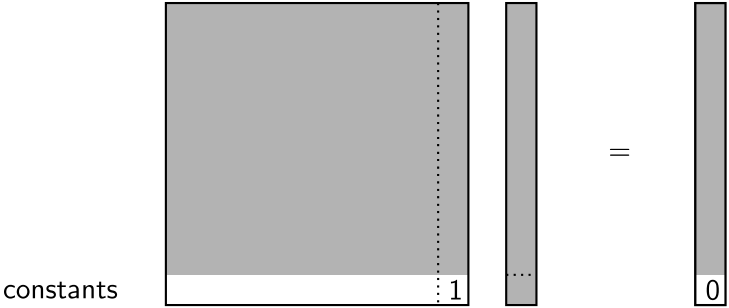

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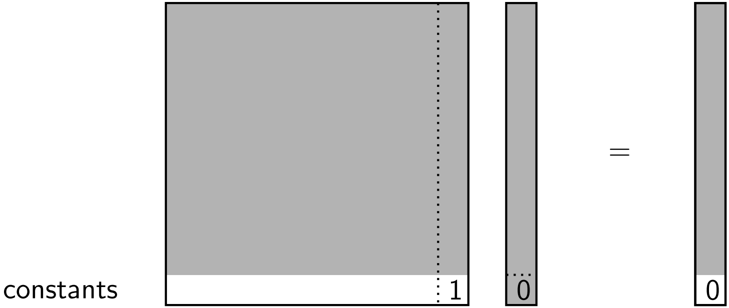

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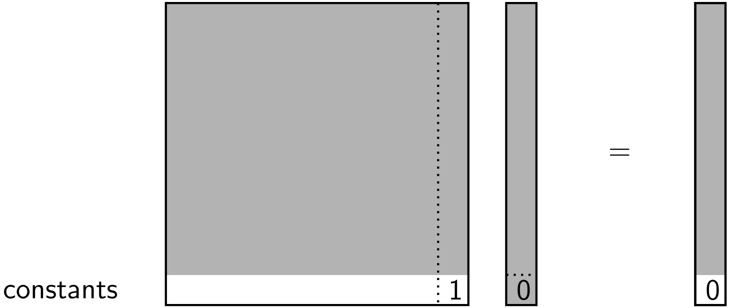

Enforced condition?

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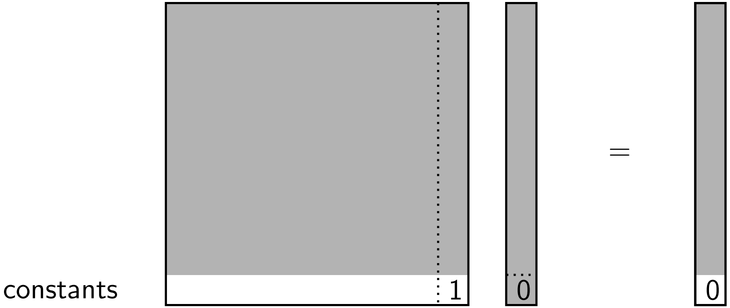

Enforced condition?

Basis vector corresponding to last column?

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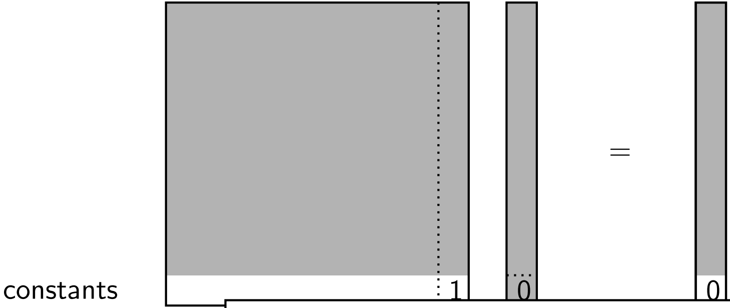

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Basis vector corresponding to last column?

Still compact?

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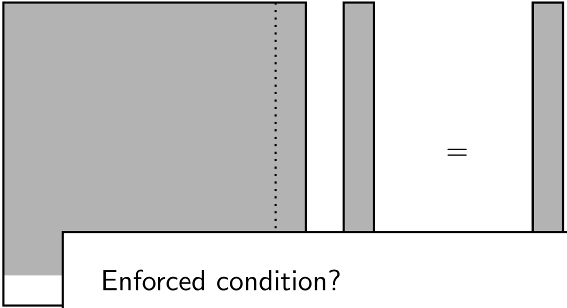

constants

Enforced condition?

Basis vector corresponding to last column?

Still compact?

Computes right BVP solution?

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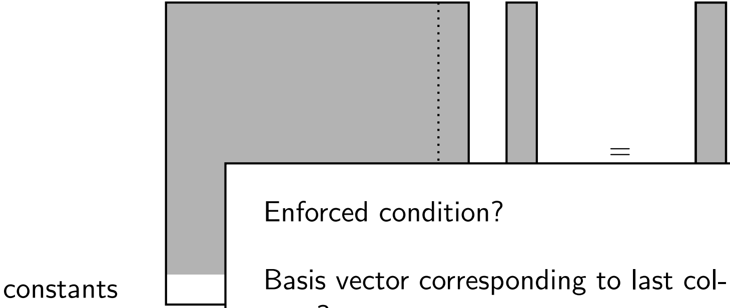

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Computes right BVP solution?

Nullspace of resulting matrix?

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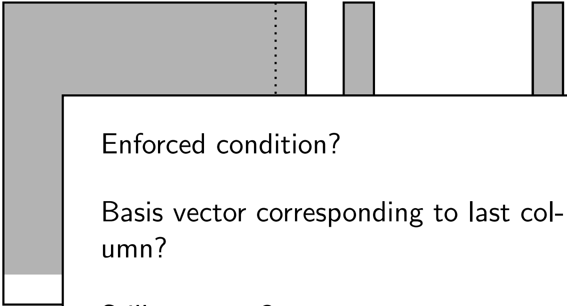

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Computes right BVP solution?

Nullspace of resulting matrix?

Conditioning of resulting matrix?

Outline

IE discretization: Nyström

IE discretization: Overview

$$\varphi(x) - \int_\Gamma K(x,y)\varphi(y)dy = f(x)$$

Nyström

- Approximate integral by quadrature: $\int_\Gamma f(y)dy \to \sum_{k=1}^{n} \omega_k f(y_k)$
- Evaluate quadrature'd IE at quadrature nodes, solve

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Projection

- Consider residual: $R := \varphi - A\varphi - f$
- Pick projection P_n (with $P_n^2 = P$) onto finite-dim. subspace
  $P_n\varphi := \sum_{k=1}^n \langle \varphi, v_k\rangle w_k \quad \to \quad$ DOFs $\langle \varphi, v_k\rangle$
- Solve $P_n R = 0$

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Important in projection methods:
*sub*space (e.g. of $C(\Gamma)$)

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- Galerkin: $v_k = w_k$
- Collocation:
  $v_k = \delta(x_k)$, $w_k(x_j) = \delta_{jk}$
- Petrov-Galerkin: $v_k \neq w_k$

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Collocation and Nyström: the same?

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1. Approximate integral by quadrature:

$$\varphi_n(x) - \sum_{k=1}^{n} \omega_k K(x, y_k)\varphi_n(y_k) = f(x) \qquad (1)$$

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2. Evaluate quadrature'd IE at quadrature nodes, solve discrete system

$$\varphi_j^{(n)} - \sum_{k=1}^{n} \omega_k K(x_j, y_k)\varphi_k^{(n)} = f(x_j) \tag{2}$$

with $x_j = y_j$ and $\varphi_j^{(n)} = \varphi_n(x_j) = \varphi_n(y_j)$

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(2) $\Rightarrow$ (1)?
Empty statement?

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I.e.: does the method work at all?

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Can stay in function space, no need to mess with varying dimensionality.

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