Integral Equations and Fast Algorithms
Lecture 22: Nyström error analysis

CS 598AK · November 7, 2013

Outline

IE discretization: Nyström

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$$\varphi_j^{(n)} - \sum_{k=1}^{n} \omega_k K(x_j, y_k)\varphi_k^{(n)} = f(x_j) \tag{2}$$

with $x_j = y_j$ and $\varphi_j^{(n)} = \varphi_n(x_j) = \varphi_n(y_j)$

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Can stay in function space, no need to mess with varying dimensionality.

Convergence for Nyström?

Increase number of quadrature points n:
Get sequence (A_n)

Want $A_n \to A$ in some sense

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No: Pick $\psi_\varepsilon = 1$ everywhere except in ε-nbh of quad nodes, 0 there.
Show:

- $\|A\varphi\psi_\varepsilon - A\varphi\|_\infty \to 0$ $(\varepsilon \to 0)$
- $\|A - A_n\|_\infty \geq \|A\|_\infty$

Compactness-based convergence

X Banach space (think: of functions)

Theorem (Not-quite-norm convergence [Kress LIE Cor 10.4])

$A_n : X \to X$ bounded linear operators,
functionwise convergent to $A : X \to X$
Then convergence is uniform on compact subsets $U \subset X$, i.e.

$$\sup_{\varphi \in U} \|A_n \varphi - A\varphi\| \to 0 \qquad (n \to \infty)$$

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Set $\mathcal{A}$ of operators $A : X \to X$

Definition (Collectively compact)

$\mathcal{A}$ is called *collectively compact* iff
for $U \subset X$ bounded, $\mathcal{A}(U)$ is relatively compact.

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How can we use the two together?

Making use of collective compactness

X Banach space, $A_n : X \to X$, (A_n) collectively compact, $A_n \to A$ functionwise.

Just showed

Corollary (Post-compact convergence [Kress LIE Cor 10.8])

- $\|(A_n - A)A\| \to 0$
- $\|(A_n - A)A_n\| \to 0$

$(n \to \infty)$

Making use of collective compactness

Assume $(I - A)^{-1}$ exists, with A compact, (A_n) collectively compact and $A_n \to A$ functionwise.

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Let $\varphi - A\varphi = f$ and $\varphi_n - A_n\varphi_n = f_n$.
$(I - A_n)(\varphi_n - \varphi) = ?$
Derive error estimate.

Questions?

?