Because the lecture video recording does not properly show the whiteboard in today's recording, this document provides a written record of the two main derivations we did on the board during lecture 23. Note that I won't try to reproduce the whole lecture content in here, just the missing derivations. Sorry for the inconvenience, I'll try to make sure this won't happen again.

Derivation 1: Anselone's theorem (slide 5 of the slides PDF, pages 15–25)

We started by noticing

$$I+(I-A)^{-1}A=(I-A)^{-1}$$

and defined

$$B_n:=I+(I-A)^{-1}A_n$$

as an approximate inverse for $(I-A_n)$. We had shown last time that

$$B_n(I-A_n)=I-S_n, \tag{1}$$

where

$$S_n:=(I-A)^{-1}(A_n-A)A.$$

Recall we're assuming that the A_n are collectively compact. From the 'post-compact convergence' corollary (slide 4, page 14), we see that $\|S_n\|\to 0$ as $n\to\infty$. Therefore there exists an n such that $\|S_n\|<1$, so that the Neumann series tells us that $(I-S_n)^{-1}$ exists, and by (1), we find that $I-A_n$ must also be invertible.

By rearranging (1), we find

$$(I-A_n)^{-1}=(I-S_n)^{-1}B_n,$$

and the Neumann series gives us

$$\|(I-S_n)^{-1}\|\leqslant\frac{1}{1-\|S_n\|}$$

Let φ be the exact density that solves $(I-A)\varphi=f$ and φ_n the approximate density that solves $(I-A_n)\varphi_n=f_n$. Then consider

$$\begin{aligned}(I-A_n)(\varphi_n-\varphi) &= f_n-(I-A_n)\varphi\\ &= f_n-(I-A+A-A_n)\varphi\\ &= f_n-f+(A-A_n)\varphi\end{aligned}$$

Combining all this knowledge as

$$\begin{aligned}\|\varphi_n-\varphi\| &\leqslant \|(I-A_n)^{-1}\|(\|f_n-f\|+\|(A_n-A)\varphi\|)\\ &\leqslant \frac{\|B_n\|}{1-\|S_n\|}(\|f_n-f\|+\|(A_n-A)\varphi\|)\end{aligned}$$

gives the desired estimate.

Derivation 2: Nyström-discretized operators are collectively compact (goes with slide 7, pages 30–34)

We use Arzelà-Ascoli to show collective compactness. To do so, we'll need to show uniform boundedness (easy!) and equicontinuity of the sequence $(A_n\varphi)$ for a given density φ. To show the latter, consider

$$
\begin{aligned}
& |(A_n\varphi)(x_1) - (A_n\varphi)(x_2)| \\
= & \left|\sum_{i=1}^{n} \omega_i (K(x_1, y_i)\varphi(y_i) - K(x_1, y_i)\varphi(y_i))\right| \\
\leqslant & \sum_{i=1}^{n} |\omega_i| \underbrace{(K(x_1, y_i) - K(x_1, y_i))}_{(*)} \|\varphi\|_\infty.
\end{aligned}
$$

We can bound $(*)$ the same way we did when we showed that continuous kernels (on compact domains!) give rise to compact operators, using the fact that K is uniformly continuous on such domains. The only term that remains a problem is

$$\sum_{i=1}^{n} |\omega_i|,$$

so if we assume

$$\sum_{i=1}^{n} |\omega_i| \leqslant C$$

independent of n, we obtain an n-independent estimate for the difference above, and thereby obtain collective compactness.

Derivation 3: Céa's Lemma

(We'll redo this next time since it ran into the end of the lecture anyhow.)