Integral Equations and Fast Algorithms
Lecture 24: Projection error, Intro to quadrature

CS 598AK · November 14, 2013

Outline

IE discretization: Projection

Singular quadrature

Céa's Lemma: Convergence for Projection

X Banach spaces, $A : X \to X$ injective, $P_n : X \to X_n$

Theorem (Céa's Lemma [Kress LIE Thm 13.6])

Convergence of the projection method
$\Leftrightarrow$ *There exist n_0 and M such that for $n \geq n_0$*

1. $P_n A : X_n \to X_n$ *are invertible,*
2. $\|(P_n A)^{-1} P_n A\| \leq M$.

In this case,

$$\|\varphi_n - \varphi\| \leq (1 + M) \inf_{\psi \in X_n} \|\varphi - \psi\|$$

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- Build error estimate.

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Core message of the theorem?

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Exact projection methods: hard. (Why?)
What if we implement a perturbation?

Perturbations of projection methods

- $A : X \to X$ bounded linear operator
- with bounded inverse
- Projection method converges for A (Céa)
- $B : X \to X$ bounded linear 'perturbation' operator

Theorem (Perturbations of projection methods [Kress LIE Thm 13.7])

If

- $\|P_n B|_{X_n}\| \to 0$ $(n \to 0)$, **or**
- B *compact and* $A + B$ *has no nullspace*

then

the projection method still converges for $A + B$.

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Ingredients to showing that a projection (e.g. Galerkin) method works?

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Ingredients to showing that a projection (e.g. Galerkin) method works?

Compare and contrast with Nyström.

Which to use?

Quote Kress LIE, 2nd ed., p. 244 (Sec. 14.1):

> *[. . .] the Nyström method is generically stable whereas the collocation and Galerkin methods may suffer from instabilities due to a poor choice of basis for the approximating subspace.*

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Quote Kress LIE, 2nd ed., p. 244 (Sec. 13.5):

> *In principle, for the Galerkin method for equations of the second kind the same remarks as for the collocation method apply. As long as numerical quadratures are available, in general, the Galerkin method cannot compete in efficiency with the Nyström method. Compared with the collocation method, it is less efficient, since its matrix elements require double integrations.*

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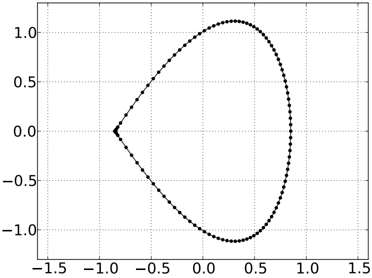
Remaining advantage of Galerkin:
Can be made not to break for non-second-kind.

Galerkin without the pain



Problem: Singular behavior at corner points.

Density may blow up.

[Bremer et al. '11]

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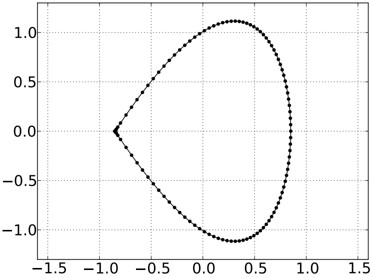

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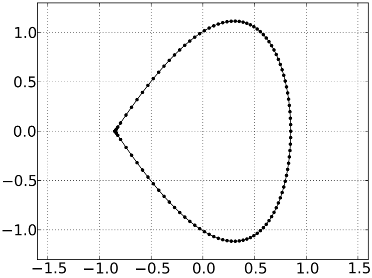

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Conditioning of the discrete system?

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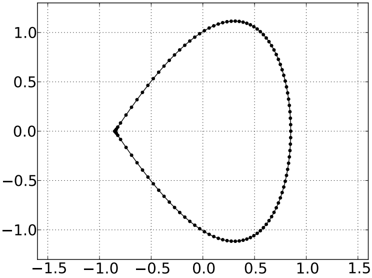

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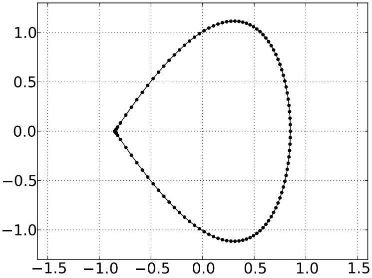

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GMRES will flail and break, because it sees $\ell^2 \sim l^\infty \sim L^\infty$ convergence.

Make GMRES 'see' L^2 convergence by redefining density DOFs:

$$\overline{\boldsymbol{\sigma}}_h := \begin{pmatrix} \sqrt{\omega_1}\sigma(x_1) \\ \vdots \\ \sqrt{\omega_n}\sigma(x_n) \end{pmatrix} = \sqrt{\boldsymbol{\omega}}\boldsymbol{\sigma}_h$$

So $\overline{\boldsymbol{\sigma}}_h \cdot \overline{\boldsymbol{\sigma}}_h = ?$

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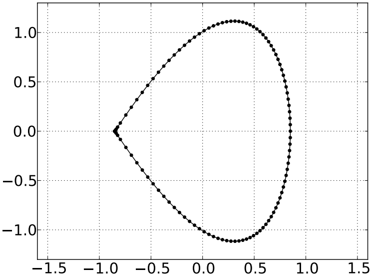

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Also fixes system conditioning! Why?

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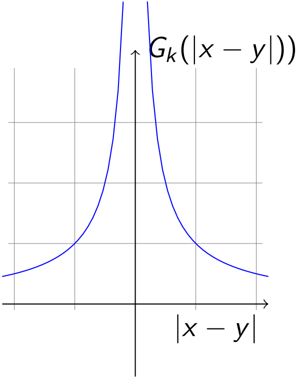
- Special-purpose methods

"Off-the-shelf" approaches

Discuss advantages, drawbacks:

- Why not Gaussian?



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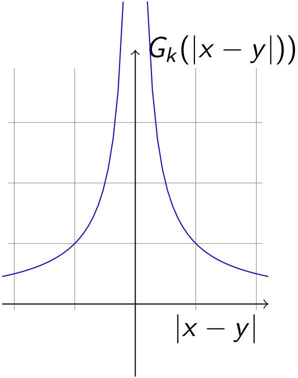

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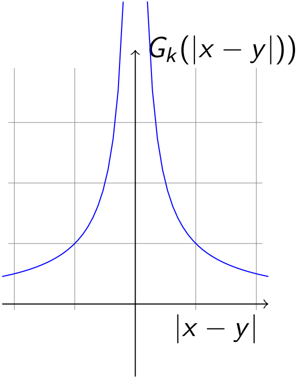
- Adaptive integration



Kussmaul/Martensen (aka “Kress”)

Theorem (A special integral [Kress LIE Lemma 8.21])

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left(4\sin^2\frac{t}{2}\right) e^{imt}dt = \begin{cases} 0 & m = 0, \\ -\frac{1}{|m|} & m = \pm 1, \pm 2 \dots . \end{cases}$$

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Exciting?

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What if you knew how to integrate
Laplace and wanted to do Helmholtz?

Singularity subtraction

$$\int \langle \text{Thing } X \text{ you would like to integrate} \rangle$$

$$= \int \langle \text{Thing } Y \text{ you \textit{can} integrate} \rangle$$

$$+ \int \langle \text{Difference } X - Y \text{ which is easy to integrate (numerically)} \rangle$$

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Drawback?

Questions?

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