Integral Equations and Fast Algorithms
Lecture 3: Functional Analysis/Intro Int.Eq.

CS 598AK · September 3, 2013

Today

Operators

Integral equations: an introduction

Outline

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Linear operators

X, Y: Banach spaces

$A : X \to Y$ linear operator

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What does linear mean here?

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Is there a notion of 'continuous at x' for linear operators?

Come up with a convenient test for boundedness.

Examples of linear operators

Examples:

- Multiplication by a scalar
- “Left shift”
- Fourier transform
- Differentiation
- Integration
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Need to know spaces to answer that!

Working with linear operators

- The set of bounded linear operators is itself a Banach space: $L(X, Y)$
- $\|Ax\| \leq \|A\|\|x\|$
- $\|BA\| \leq \|B\|\|A\|$

Greatest hits from functional analysis

- Hahn-Banach theorem
- Open mapping theorem / bounded inverse theorem
- Banach-Steinhaus / Uniform boundedness principle

Outline

Basic zoology

Volterra	Fredholm
$\int_a^x k(x,y)f(y)dy = g(x)$	$\int_G k(x,y)f(y)dy = g(x)$

First kind	Second kind
$\int_G k(x,y)f(y)dy = g(x)$	$f(x) + \int_G k(x,y)f(y)dy = g(x)$

$k(x)$: Kernel
$f(x)$: Density (unknown)
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Sensible bounds on x?

2D/3D analogs?

Factor allowable in front of identity?

Why second kind?

$$f(x) + \int_G k(x,y)f(y)dy = ((I+K)f)(x) = g(x)$$

Why even talk about 'second-kind operators'?

- Throw a $+\delta(x-y)$ into the kernel, back to looking like first kind. So?

Why second kind?

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Why even talk about 'second-kind operators'?

- Throw a $+\delta(x-y)$ into the kernel, back to looking like first kind. So?
- Is the identity in $(I+K)$ crucial?

Some intuition

$G(x, y)$: displacement of (linearly elastic) beam in z at x in response to point unit load at point y

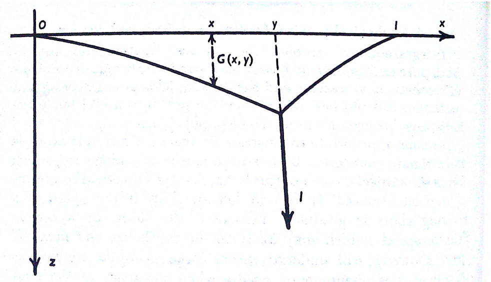

Fig. 1

[Tricomi, p.2]

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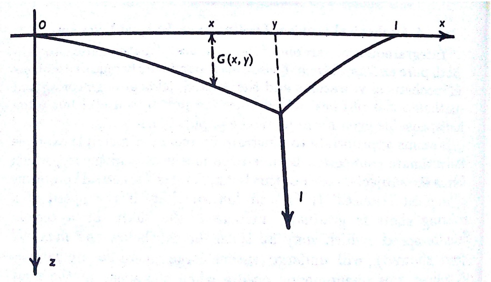

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Given load distribution $p(y)$.
Compute displacement $z(x)$.

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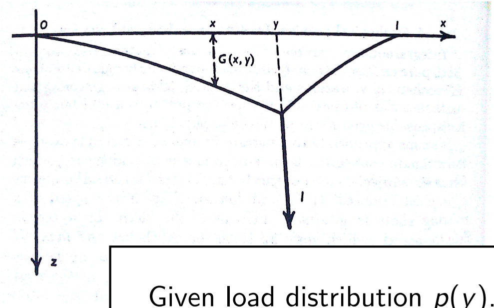

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First kind equation?

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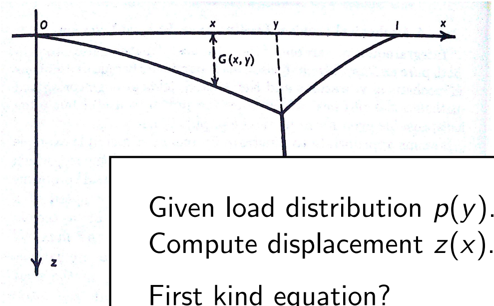

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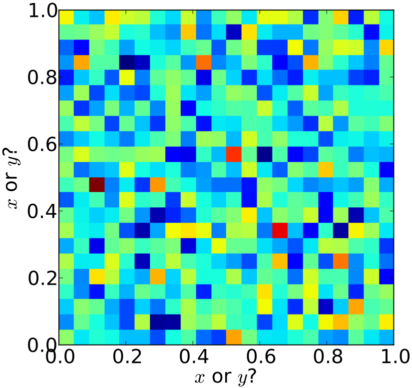
Given load distribution $p(y)$.
Compute displacement $z(x)$.

First kind equation?

Second kind equation?
Let $p(x) = \omega^2 \mu(x) z(x)$.

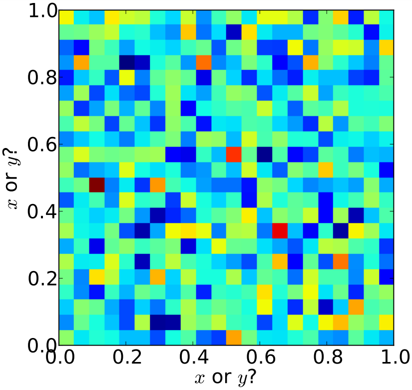
Matrix multiplication as an integral operator



Build an integral operator that (in some way) computes

$$(Ax)_i = \sum_j A_{ij} x_j$$

Matrix multiplication as an integral operator



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Knowledge about function solving the IE analog of $Ax = b$ for second-kind IE? What about first-kind?

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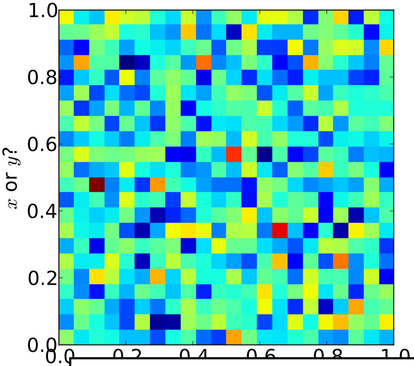

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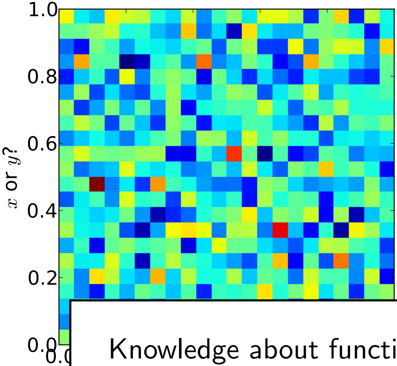

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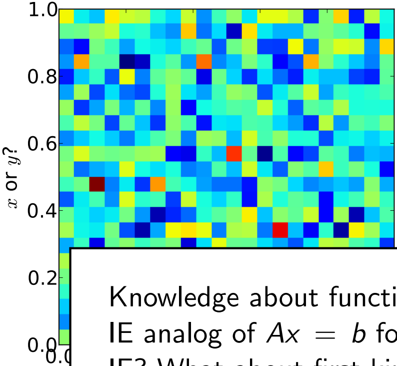

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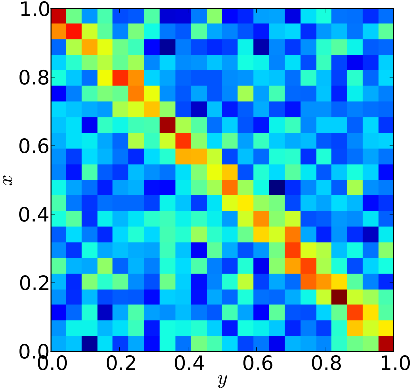

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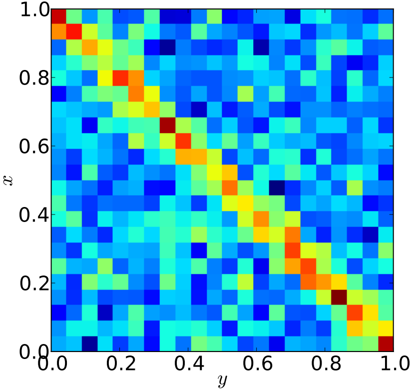

What about diagonally dominant matrices?

Linear ODE to Volterra

Linear, variable-coefficient IVP [Tricomi, 1.8]

$$u^{(n)} + a_1(x)u^{(n-1)} + \cdots + a_n(x)u = F(x)$$
$$u(0) = c_0, \quad u'(0) = c_1, \quad \ldots, \quad u^{(n-1)}(0) = c_{n-1}$$

Define $\varphi := u^{(n)}$ and then

$$D^{-1}\varphi(x) = \int_0^x \varphi(y)dy \qquad \ldots \quad D^{-n} = (D^{-1})^n$$

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Oddly enough:

$$D^{-n}\varphi(x) = \frac{1}{(n-1)!}\int_0^x (x-y)^{n-1}\varphi(y)dy$$

(Homework!)

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in terms of $D^{-i}\varphi$ and c_i?

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Result is even second-kind. (Why?)

Questions?

?