Integral Equations and Fast Algorithms
Lecture 4: Intro Int.Eq./Compactness

CS 598AK · September 5, 2013

Today

Integral equations: an introduction, cont'd.

Second-kind equations, Attempt 1: Neumann

Compactness

Outline

Linear ODE to Volterra

Linear, variable-coefficient IVP [Tricomi, 1.8]

$$u^{(n)} + a_1(x)u^{(n-1)} + \cdots + a_n(x)u = F(x)$$
$$u(0) = c_0, \quad u'(0) = c_1, \quad \ldots, \quad u^{(n-1)}(0) = c_{n-1}$$

Define $\varphi := u^{(n)}$ and then

$$D^{-1}\varphi(x) = \int_0^x \varphi(y)dy \qquad \ldots \quad D^{-n} = (D^{-1})^n$$

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Oddly enough:

$$D^{-n}\varphi(x) = \frac{1}{(n-1)!}\int_0^x (x-y)^{n-1}\varphi(y)dy$$

(Homework!)

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$$u = \cdots$$

in terms of $D^{-i}\varphi$ and c_i?

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Result is even second-kind. (Why?)

Volterra: first kind

$$\int_a^x K(x,y)\varphi(y)dy = f(x) \tag{1}$$

Take derivative of both sides of (1). What happens?

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Complex analysis

Complex analysis is *full* of integral operators:

- Cauchy's integral formula:

$$f(a) = \frac{1}{2\pi i} \oint_\gamma \frac{1}{z-a} f(z)\, dz$$

- Cauchy's differentiation formula:

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_\gamma \frac{1}{(z-a)^{n+1}} f(z)\, dz$$

Basic Marketing Message

You may not have noticed them, ...
but integral operators/equations are *everywhere*!

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Compactness

Integral operators: continuity

Theorem (Continuous kernel ⇒ bounded)

$G \subset \mathbb{R}^n$ closed, bounded ("compact"), $K \in C(G^2)$. Let

$$(A\varphi)(x) := \int_G K(x,y)\varphi(y)dy.$$

Then

$$\|A\|_\infty = \max_{x \in G} \int_G |K(x,y)|dy.$$

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Show "$\leq$".

Making a connection

Want to solve

$$\varphi - A\varphi = (I - A)\varphi = f.$$

Formally:

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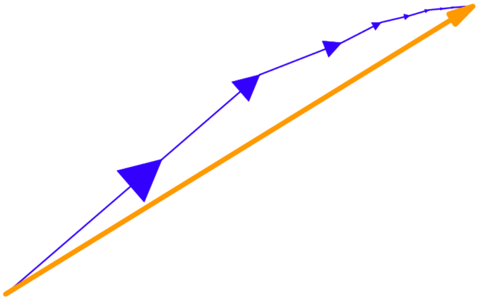
if. . . ?

Recap: completeness



Completeness means that

if a particle moves along the broken path (in blue) travelling a finite total distance, then the particle has a well-defined net displacement (in orange).

[Wikipedia]

Neumann series

Theorem

$A : X \to X$, $\|A\| < 1$

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with $\|(I - A)^{-1}\| \leq 1/(1 - \|A\|)$.

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- Show bound.
- Partial sums form Cauchy sequence.
- Limit exists. Is it the inverse? Use

$$(I - A) \sum_{k=0}^{n} A^k = (I - A^{n+1})$$

Neumann series, applied

Let partial sums (of Neumann series) be

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Then

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Lame because…(at least 2 reasons)?

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X Banach space

Definition (Precompact/Relatively compact)

$M \subseteq X$ precompact:$\Leftrightarrow$ all sequences $(x_k) \subset M$ contain a subsequence converging in X

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Precompactness 'replaces' boundedness in ∞ dim

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X, Y: Banach spaces

Definition (Compact operator)

$T : X \to Y$ is *compact* $:\Leftrightarrow$ $T(\bar{B}_1(0))$ is precompact.

Compact operators

Define $\bar{B}_1(0)$.

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- T_n all compact, $T_n \to T$ in operator norm $\Rightarrow T$ compact

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Let $\dim T(X) < \infty$. Is T compact?

Is the identity operator compact?

Questions?

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