Integral Equations and Fast Algorithms
Lecture 5: Compactness/Fredholm

CS 598AK · September 10, 2013

Discussion

- Will post list of papers for final project tonight
  - Only *suggestions*–need not pick only from list
- Trapezoidal rule weighting
  - Trapezoidal for periodic
  - `numpy.linspace`
  - Algebraic convergence vs spectral convergence
- git
- Reading from the command line
- HW interval
- "Vectorization"
- Log color scales
- String formatting
- 3D plotting

Today

Compactness

Outline

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X Banach space

Definition (Precompact/Relatively compact)

$M \subseteq X$ precompact:$\Leftrightarrow$ all sequences $(x_k) \subset M$ contain a subsequence converging in X

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Counterexample?

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Precompactness 'replaces' boundedness in ∞ dim

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$T : X \to Y$ is *compact* $:\Leftrightarrow$ T(bounded set) is precompact.

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Define $\bar{B}_1(0)$.

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- T_n all compact, $T_n \to T$ in operator norm $\Rightarrow$ T compact

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Is the identity operator compact?

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Let G be compact.

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Intuition?

"Uniformly continuous"?

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- $A(U)$ equicontinuous?
- K *uniformly* continuous on $G \times G$ because $G \times G$ compact.

Questions?

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