Integral Equations and Fast Algorithms
Lecture 6: Compactness/Fredholm

CS 598AK · September 12, 2013

Today

Compactness

Outline

Arzelà-Ascoli

Let G be compact.

Theorem (Arzelà-Ascoli)

$U \subset C(G)$ is precompact iff it is bounded and equicontinuous.

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Continuous means:

For all $x, y \in G$
for all $\varepsilon > 0$ there exists a $\delta > 0$ such that
if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

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“Uniformly continuous”?

Integral operators: compactness

Theorem (Continuous kernel $\Rightarrow$ compact [Kress LIE Thm. 2.21])

$G \subset \mathbb{R}^m$ compact, $K \in C(G^2)$. Then

$$(A\varphi)(x) := \int_G K(x,y)\varphi(y)dy.$$

is compact on $C(G)$.

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  What is there to show?

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- Pick $U \subset C(G)$. $A(U)$ bounded?
- $A(U)$ equicontinuous?
- K *uniformly* continuous on $G \times G$ because $G \times G$ compact.

Weakly singular

$G \subset \mathbb{R}^n$ compact

Definition (Weakly singular kernel)

- K defined, continuous everywhere except at $x = y$
- There exist $C > 0$, $\alpha \in (0, n]$ such that

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Theorem (Weakly singular kernel $\Rightarrow$ compact [Kress LIE Thm. 2.22])

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- A is limit of compact operators.

Questions?

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