Integral Equations and Fast Algorithms
Lecture 7: Compactness/Fredholm

CS 598AK · September 17, 2013

Today

Compactness

Second-kind equations, Attempt 2: Riesz

Outline

Compactness

Second-kind equations, Attempt 2: Riesz

Integral operators: compactness

Theorem (Continuous kernel $\Rightarrow$ compact [Kress LIE Thm. 2.21])

$G \subset \mathbb{R}^m$ compact, $K \in C(G^2)$. Then

$$(A\varphi)(x) := \int_G K(x,y)\varphi(y)dy.$$

is compact on $C(G)$.

Integral operators: compactness

Theorem (Continuous kernel $\Rightarrow$ compact [Kress LIE Thm. 2.21])

$G \subset \mathbb{R}^m$ compact, $K \in C(G^2)$. Then

$$(A\varphi)(x) := \int_G K(x,y)\varphi(y)dy.$$

is compact on $C(G)$.

- Use A-A. (a statement about compact *sets*)
  What is there to show?

Integral operators: compactness

Theorem (Continuous kernel $\Rightarrow$ compact [Kress LIE Thm. 2.21])

$G \subset \mathbb{R}^m$ compact, $K \in C(G^2)$. Then

$$(A\varphi)(x) := \int_G K(x,y)\varphi(y)dy.$$

is compact on $C(G)$.

- Use A-A. (a statement about compact *sets*)
  What is there to show?
- Pick $U \subset C(G)$. $A(U)$ bounded?

Integral operators: compactness

Theorem (Continuous kernel $\Rightarrow$ compact [Kress LIE Thm. 2.21])

$G \subset \mathbb{R}^m$ compact, $K \in C(G^2)$. Then

$$(A\varphi)(x) := \int_G K(x,y)\varphi(y)dy.$$

is compact on $C(G)$.

- Use A-A. (a statement about compact *sets*)
  What is there to show?
- Pick $U \subset C(G)$. $A(U)$ bounded?
- $A(U)$ equicontinuous?

Integral operators: compactness

Theorem (Continuous kernel $\Rightarrow$ compact [Kress LIE Thm. 2.21])

$G \subset \mathbb{R}^m$ compact, $K \in C(G^2)$. Then

$$(A\varphi)(x) := \int_G K(x,y)\varphi(y)dy.$$

is compact on $C(G)$.

- Use A-A. (a statement about compact *sets*)
  What is there to show?
- Pick $U \subset C(G)$. $A(U)$ bounded?
- $A(U)$ equicontinuous?
- K *uniformly* continuous on $G \times G$ because $G \times G$ compact.

Weakly singular

$G \subset \mathbb{R}^n$ compact

Definition (Weakly singular kernel)

- K defined, continuous everywhere except at $x = y$
- There exist $C > 0$, $\alpha \in (0, n]$ such that

$$|K(x, y)| \leq C|x - y|^{\alpha - n} \qquad (x \neq y)$$

Weakly singular

$G \subset \mathbb{R}^n$ compact

Definition (Weakly singular kernel)

- K defined, continuous everywhere except at $x = y$
- There exist $C > 0$, $\alpha \in (0, n]$ such that

$$|K(x, y)| \leq C|x - y|^{\alpha - n} \qquad (x \neq y)$$

Theorem (Weakly singular kernel $\Rightarrow$ compact [Kress LIE Thm. 2.22])

K weakly singular. Then

$$(A\varphi)(x) := \int_G K(x, y)\varphi(y)dy.$$

is compact on $C(G)$.

Weakly singular

$G \subset \mathbb{R}^n$ compact

Definition (Weakly singular kernel)

- K defined, contin
- There exist $C > 0$

$$|K(x$$

Theorem (Weakly sin

K weakly singular. Th

$$(A\varphi$$

is compact on $C(G)$.

- Show boundedness/existence as improper integral. (polar coordinates)

Weakly singular

$G \subset \mathbb{R}^n$ compact

Definition (Weakly singular kernel)

- K defined, contin
- There exist $C > ($

$$|K(x$$

Theorem (Weakly sin

K weakly singular. Th

$$(A\varphi$$

is compact on $C(G)$.

- Show boundedness/existence as improper integral. (polar coordinates)
- Bleep out the singularity with a C^0 PoU that shrinks with n: A_n

Weakly singular

$G \subset \mathbb{R}^n$ compact

Definition (Weakly singular kernel)

- K defined, contin
- There exist $C > ($

$|K(x$

Theorem (Weakly sin

K weakly singular. Th

$(A\varphi$

is compact on $C(G)$.

- Show boundedness/existence as improper integral. (polar coordinates)
- Bleep out the singularity with a C^0 PoU that shrinks with n: A_n
- Each A_n compact by previous thm.

Weakly singular

$G \subset \mathbb{R}^n$ compact

Definition (Weakly singular kernel)

- K defined, contin
- There exist $C > ($

$$|K(x$$

Theorem (Weakly sir

K weakly singular. Th

$$(A\varphi$$

is compact on $C(G)$.

- Show boundedness/existence as improper integral. (polar coordinates)
- Bleep out the singularity with a C^0 PoU that shrinks with n: A_n
- Each A_n compact by previous thm.
- Shrink singularity with n. A_n converge uniformly (because of weak singularity).

Weakly singular

$G \subset \mathbb{R}^n$ compact

Definition (Weakly singular kernel)

- K defined, contin
- There exist $C > 0$

$|K(x$

Theorem (Weakly sir

K weakly singular. Th

$(A\varphi$

is compact on $C(G)$.

- Show boundedness/existence as improper integral. (polar coordinates)
- Bleep out the singularity with a C^0 PoU that shrinks with n: A_n
- Each A_n compact by previous thm.
- Shrink singularity with n. A_n converge uniformly (because of weak singularity).
- A is limit of compact operators.

Weakly singular (on surfaces)

$\Omega \subset \mathbb{R}^n$ bounded, open, C^1

Definition (Weakly singular kernel)

- K defined, continuous everywhere except at $x = y$
- There exist $C > 0$, $\alpha \in (0, n-1]$ such that

$$|K(x,y)| \leq C|x-y|^{\alpha-n+1} \qquad (x, y \in \partial\Omega,\ x \neq y)$$

Weakly singular (on surfaces)

$\Omega \subset \mathbb{R}^n$ bounded, open, C^1

Definition (Weakly singular kernel)

- K defined, continuous everywhere except at $x = y$
- There exist $C > 0$, $\alpha \in (0, n-1]$ such that

$$|K(x,y)| \leq C|x-y|^{\alpha-n+1} \qquad (x, y \in \partial\Omega,\ x \neq y)$$

Has this estimate gotten more or less strict?

Weakly singular (on surfaces)

$\Omega \subset \mathbb{R}^n$ bounded, open, C^1

Definition (Weakly singular kernel)

- K defined, continuous everywhere except at $x = y$
- There exist $C > 0$, $\alpha \in (0, n-1]$ such that

$$|K(x,y)| \leq C|x-y|^{\alpha-n+1} \qquad (x, y \in \partial\Omega,\ x \neq y)$$

Weakly singular (on surfaces)

$\Omega \subset \mathbb{R}^n$ bounded, open, C^1

Definition (Weakly singular kernel)

- K defined, continuous everywhere except at $x = y$
- There exist $C > 0$, $\alpha \in (0, n-1]$ such that

$$|K(x,y)| \leq C|x-y|^{\alpha-n+1} \qquad (x, y \in \partial\Omega,\ x \neq y)$$

Theorem (Weakly singular kernel $\Rightarrow$ compact [Kress LIE Thm. 2.23])

K weakly singular on $\partial\Omega$. Then

$$(A\varphi)(x) := \int_G K(x,y)\varphi(y)dy.$$

is compact on $C(G)$.

Outline

Compactness

Second-kind equations, Attempt 2: Riesz

Riesz Theory (I)

Again, trying to solve

$$L\varphi := (I - A)\varphi = \varphi - A\varphi = f$$

with A compact.

Riesz Theory (I)

Again, trying to solve

$$L\varphi := (I - A)\varphi = \varphi - A\varphi = f$$

with A compact.

Theorem (First Riesz Theorem [Kress, Thm. 3.1])

$N(L)$ is finite-dimensional.

Riesz Theory (I)

Again, trying to solve

$$L\varphi := (I - A)\varphi = \varphi - A\varphi = f$$

with A compact.

Theorem (First Riesz Theorem [Kress, Thm. 3.1])

$N(L)$ is finite-dimensional.

- What is $N(L)$ again?
- Why is this good news?

Riesz Theory (I)

Again, trying to solve

$$L\varphi := (I - A)\varphi = \varphi - A\varphi = f$$

with A compact.

Theorem (First Riesz Theorem [Kress, Thm. 3.1])

$N(L)$ is finite-dimensional.

Riesz Theory (I)

Again, trying to solve

$$L\varphi := (I - A)\varphi = \varphi - A\varphi = f$$

with A compact.

Theorem (First Riesz Theorem [Kress, Thm. 3.1])

$N(L)$ is finite-dimensional.

- $N(L)$ closed. (Why?)

Riesz Theory (I)

Again, trying to solve

$$L\varphi := (I - A)\varphi = \varphi - A\varphi = f$$

with A compact.

Theorem (First Riesz Theorem [Kress, Thm. 3.1])

$N(L)$ is finite-dimensional.

- $N(L)$ closed. (Why?)
- $L\varphi = 0$ means what for A?

Riesz Theory (I)

Again, trying to solve

$$L\varphi := (I - A)\varphi = \varphi - A\varphi = f$$

with A compact.

Theorem (First Riesz Theorem [Kress, Thm. 3.1])

$N(L)$ is finite-dimensional.

- $N(L)$ closed. (Why?)
- $L\varphi = 0$ means what for A?
- When is the identity compact again?

Questions?

?