[A note on the rectangle rule.] I just watched the beginning of the lecture and spotted a dumb mistake. The area of the error under one segment in the rectangle rule doesn't scale with h as I said in class. It scales with h^2. (Easy to see if you think about it–there are two sides to the triangle, the area is $\Delta f \cdot h/2$, and $\Delta f \sim h\, f'$ by Taylor. The fact that there are $N = 1/h$ contributions to the error brings the overall error to h. Sorry about that. :)

[(part of) Kress, LIE, Thm. 3.10] To show: the homogeneous second-kind Volterra integral equation

$$(\varphi - A\varphi)(x) := \varphi(x) - \int_a^x K(x,y)\varphi(y)\mathrm{d}y = 0$$

for continuous K has only the trivial solution φ.

Proof.

Suppose we have a nontrivial solution $\varphi \neq 0$ of the IE. Then $\varphi = A\varphi$. We'll show by induction:

$$\left|\varphi(x)\right| \leqslant \|\varphi\|_\infty \frac{M^n (x-a)^n}{n!},$$

where $M := \|K\|_\infty$.

For $n = 0$, we get

$$|\varphi(x)| \leqslant \|\varphi\|_\infty,$$

which is true. Then assume that the induction claim is true for n. We'll now show that it's true for $n+1$.

$$\begin{aligned}
|\varphi(x)| &= \left|\int_a^x K(x,y)\varphi(y)\mathrm{d}y\right| \\
&\leqslant \int_a^x |K(x,y)\varphi(y)|\mathrm{d}y \\
\text{Use induction claim for } n. \quad &\leqslant \|\varphi\|_\infty \frac{M^n}{n!}\int_a^x |(y-a)^n||K(x,y)|\mathrm{d}y \\
&\leqslant \|\varphi\|_\infty \frac{M^{n+1}}{n!}\int_a^x |(y-a)^n|\mathrm{d}y \\
&= \|\varphi\|_\infty \frac{M^{n+1}}{(n+1)!}(x-a)^{n+1}.
\end{aligned}$$