Integral Equations and Fast Algorithms
Lecture 8: Compactness/Fredholm

CS 598AK · September 19, 2013

Today

Second-kind equations, Attempt 2: Riesz

Second-kind equations, Attempt 3: Fredholm

Outline

Riesz Theory (I)

Again, trying to solve

$$L\varphi := (I - A)\varphi = \varphi - A\varphi = f$$

with A compact.

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Theorem (First Riesz Theorem [Kress, Thm. 3.1]**)**

$N(L)$ is finite-dimensional.

- What is $N(L)$ again?
- Why is this good news?

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- $L\varphi = 0$ means what for A?
- When is the identity compact again?

Riesz Theory (II)

Theorem (Riesz theory [Kress, Thm. 3.4])

A compact. **Then:**

- $(I - A)$ *injective* $\Leftrightarrow$ $(I - A)$ *surjective*
  - *It's either bijective or neither s nor i.*
- *If* $(I - A)$ *is bijective,* $(I - A)^{-1}$ *is bounded.*

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- Rephrase for solvability
- Main impact?
- Key shortcoming?

Applying Riesz to Volterra

Theorem (Volterra with continuous kernel)

$f \in C([a,b])$, $K \in C(\{y < x\})$. **Then:**

$$\varphi(x) - \int_a^x K(x,y)\varphi(y)dy = f(y)$$

always has a unique solution $\varphi \in C([a,b])$

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- Let φ solve $\varphi = A\varphi$.
- Induction: $|\varphi(x)| \leq \|\varphi\|_\infty M^n (x - a)^n / n!$

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Recap: Hilbert spaces

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Name a Hilbert space of functions.

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- Compactness results *do* transfer over. [Thm. 4.11, Pb. 4.5 in Kress LIE]

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What's this called?

Adjoint operators

Definition (Adjoint oeprator)

A^* called adjoint to A if

$$(Ax, y) = (x, A^* y)$$

for all x, y.

- A^* unique
- A^* exists
- A^* linear
- A bounded $\Rightarrow$ A^* bounded
- A compact $\Rightarrow$ A^* compact

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(in matrix representation)

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Adjoint of the single-layer?

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(in matrix representation)

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Adjoint of the single-layer?

Adjoint of the double-layer?

Questions?

?