Integral Equations and Fast Algorithms
Lecture 9: Fredholm, Spectral Theory

CS 598AK · September 24, 2013

Today

Second-kind equations, Attempt 3: Fredholm

Outline

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Adjoint operators

Definition (Adjoint oeprator)

A^* called adjoint to A if

$$(Ax, y) = (x, A^* y)$$

for all x, y.

- A^* unique
- A^* exists
- A^* linear
- A bounded $\Rightarrow$ A^* bounded
- A compact $\Rightarrow$ A^* compact

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What do you expect to happen with integral operators?

Adjoint of the single-layer?

Adjoint of the double-layer?

Fredholm Alternative

Theorem (Fredholm Alternative [Kress LIE Thm. 4.14])

$A : X \to X$ *compact.*

Then either:

- $I - A$ *and* $I - A^*$ *are bijective*

or:

- $\dim N(I - A) = \dim N(I - A^*)$
- $(I - A)(X) = N(I - A^*)^\perp$
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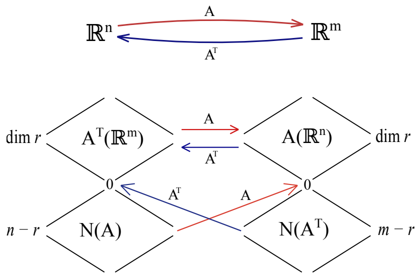
Seen these statements before?

Fredholm



[Wikipedia]

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Where to get uniqueness?

Questions?

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