Today

Second-kind equations, Attempt 3: Fredholm
Outline

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**Definition (Adjoint operator)**

$A^*$ called adjoint to $A$ if

$$(Ax, y) = (x, A^*y)$$

for all $x, y$.

- $A^*$ unique
- $A^*$ exists
- $A^*$ linear
- $A$ bounded $\Rightarrow A^*$ bounded
- $A$ compact $\Rightarrow A^*$ compact
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Adjoint operator in finite dimensions? (in matrix representation)
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What do you expect to happen with integral operators?
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Adjoint of the single-layer?
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Adjoint of the single-layer?

Adjoint of the double-layer?
Fredholm Alternative

Theorem (Fredholm Alternative [Kress LIE Thm. 4.14])

\( A : X \to X \) compact.

Then either:

- \( I - A \) and \( I - A^* \) are bijective

or:

- \( \dim N(I - A) = \dim N(I - A^*) \)
- \( (I - A)(X) = N(I - A^*)^\perp \)
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Seen these statements before?
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Translate to language of integral equation solvability
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Translate to language of integral equation solvability

Rephrase for symmetric kernels

\( (K(x, y) = K(y, x)) \)
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Rephrase for symmetric kernels \((K(x, y) = K(y, x))\)

Most-used direction:
Unique (hom) \(\Rightarrow\) Existence (inhom)
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Rephrase for symmetric kernels

$(K(x, y) = K(y, x))$

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Unique (hom) $\Rightarrow$ Existence (inhom)

Where to get uniqueness?
Questions?