Using the Vandermonde matrix for point interpolation

Let

$$V := \begin{pmatrix} \varphi_0(x_0) & \varphi_1(x_0) & \cdots & \varphi_n(x_0) \\ \varphi_0(x_1) & \varphi_1(x_1) & \cdots & \varphi_n(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_0(x_n) & \varphi_1(x_n) & \cdots & \varphi_n(x_n) \end{pmatrix}$$

be the (generalized) Vandermonde matrix for a given basis and node set.

Then, the interpolation coefficients $\boldsymbol{\alpha} = (\alpha_i)$ that perform interpolation of function values of a function f from the nodal values at x_i to a new point x (not part of the original nodal set) by

$$f(x) \approx \sum_{i=0}^{n} \alpha_i f(x_i)$$

can be found by solving

$$V^T \boldsymbol{\alpha} = \begin{pmatrix} \varphi_0(x_0) & \varphi_0(x_1) & \cdots & \varphi_0(x_n) \\ \varphi_1(x_0) & \varphi_1(x_1) & \cdots & \varphi_1(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_n(x_0) & \varphi_n(x_1) & \cdots & \varphi_n(x_n) \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \varphi_0(x) \\ \varphi_1(x) \\ \vdots \\ \varphi_n(x) \end{pmatrix}$$

for $\boldsymbol{\alpha}$. This works because if you look at the above system row-by-row, the (α_i) are determined so that they do exactly that (point interpolation) on the basis (φ_i), i.e. the ith row of the above system corresponds to

$$\varphi_i(x) = \sum_{j=0}^{n} \alpha_j \varphi_i(x_j).$$