$$A = \begin{pmatrix} 0.1 & \\ & 30 \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} 10 & \\ & \frac{1}{30} \end{pmatrix}$$

$$K_2 = \underbrace{\|A\|_2}_{30} \cdot \underbrace{\|A^{-1}\|_2}_{10} = 300$$

input $x$ output $y$

$$\underbrace{\frac{\|\Delta y\|}{\|y\|}}_{} \leq \underbrace{\text{rel.cond}}_{300} \cdot \underbrace{\frac{\|\Delta x\|}{\|x\|}}_{\text{"5 digits" } 10^{-5}}$$

| | | | |
|---|---|---|---|
| rel. err $\sim 10^{-4}$ | abs. err. .96 | < 12,374.96 | |
| | | 12,374 | true |
| | | .96 | |
| rel. err $10^{-4}$ | abs. err 13.74 | 3,141,597.74 | |
| | | 3,141,584 | true |

$$1 = \|I\| = \|AA^{-1}\| \leq \overbrace{\|A\| \, \|A^{-1}\|}^{\text{cond}}$$

$$\|AB\| \leq \|A\| \cdot \|B\|$$

$$\|Ax\| \le \|A\| \, \|x\|$$

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

$$\|x\| \, \|A\| \ge \|Ax\|$$

## Condition numbers, cont'd

- $\kappa(A^{-1}) = \|(A^{-1})^{-1}\| \cdot \|A^{-1}\| = \kappa(A)$
- $A\overset{\text{out}}{x} = \overset{\text{in}}{b}$ $\qquad$ $A^{-1}\overset{\text{in}}{b} = \overset{\text{out}}{x}$

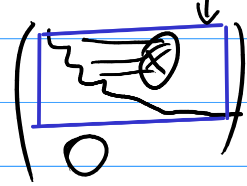
Demo

## Applications of LU/GE

- Find a basis of a span

$v_1 \dots v_{117}$

$$\begin{pmatrix} - v_1 - \\ \vdots \\ - v_{117} - \end{pmatrix} \rightsquigarrow$$



rows span the same space as $v_1 \dots v_{117}$

RREF

- Solve a system of linear equations

Have $PA = LU$ ← triangular (because $A$ invertbl.)

triangular

$n^3$

$A = P^T LU$

$P^{-1}$

Want to solve: $Ax_1 = b_1$ $\qquad$ $Ax_2 = b_2$

$P^T \underbrace{L \overbrace{Ux}^{y}}_{z} = b$

$P^T z = b \rightarrow z = Pb$ $\qquad$ $z_2 = Pb_2$ linear

$Ly = z$ (fw subst.) $\qquad$ $Ly_2 = z_2$ $n^2$

$Ux = y$ (back subst.) $\qquad$ $Ux_2 = y_2$ $n^2$

- cheaper
- can amortize the cost of factorization over multiple RHS

- Find determinant

$\det(AB) = \det(A)\det(B)$

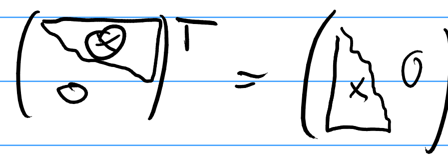
$PA = LU$ $\qquad$ $\det A = \dfrac{\det(L)\det(U)}{\det(P)}$

1 $\qquad$ product of diagonal

... (cheap)

- Find rank (but: subtle)
- Find a basis of $N(A)$

$PA = LU$ $\qquad$ $N(PA) = N(U)$



$\tilde{P}A^T = \tilde{L}\tilde{U}$

$A\tilde{P}^T = \tilde{U}^T\tilde{L}^T$

$A = \tilde{U}^T\tilde{L}^T\tilde{P}$