Inner product: (x, y)

$$(\alpha x + \beta y, z) = \alpha (x, z) + \beta (y, z)$$

$$(x, y) = (y, x)$$

$$(x, x) \geq 0$$

$$(x, x) = 0 \Leftrightarrow x = 0$$

$$M_3 M_2 M_1 A = U$$

$$\rightsquigarrow \quad A = \underbrace{(M_1^{-1} M_2^{-1} M_3^{-1})}_{L} U$$

$$\| \cdot \| = \sqrt{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}} = \sqrt{x_1^2 + \dots + x_n^2}$$

<u>Pythagorean theorem</u>

$$x \perp y \Rightarrow \|x + y\|^2 = \|x\|^2 + \|y\|^2$$

Making x and y perpendicular

$$y^{\parallel} = \frac{(x,y)}{\cancel{(x,x)}} x$$

$$y^{\perp} = y - y^{\parallel}$$

What if $\|x\| = 1$?

norm matching the inner product

Orthogonal basis $(b_i) \subseteq \underbrace{V \setminus \{0\}}_{\text{V, but without the null vector}}$ iff

"set minus"

- $\text{span}(\{b_1, \dots, b_n\}) = V$
- $b_i \perp b_j$ for $i \neq j$

"pairwise orthogonal"

implies lin. indep.

Orthonormal basis (ONB):

- orthogonal basis
- $\|b_i\| = 1$

Have: $x \in V$, (b_i) ONB

Want: $\underbrace{(x, b_1)}_{\alpha_1} b_1 + \underbrace{(x, b_2)}_{\alpha_2} b_2 + \dots + \underbrace{(x, b_n)}_{\alpha_n} b_n \overset{?}{=} x$

$\longrightarrow$ HW

In the non-orth case:

$$\begin{pmatrix} | & | & | & & | \\ b_1 & b_2 & b_3 & \cdots & b_n \\ | & | & | & & | \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = x$$

Now: choose the dot prod as our inner prod.

Want to find coefficients by multiplying by a matrix

$$\underbrace{\begin{pmatrix} — b_1 — \\ — b_2 — \\ \vdots \\ — b_n — \end{pmatrix}}_{Q^T} \begin{pmatrix} x \end{pmatrix} = \begin{pmatrix} x \cdot b_1 \\ x \cdot b_2 \\ \vdots \\ x \cdot b_n \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

Orthogonal matrix: iff it has orthonormal basis for colums

• Suppose we have an orthogonal matrix Q

$$QQ^T = Id$$

$Q^T x =$ coefficients

- Is Q square? yes
- Is Q^T orthogonal aswell?

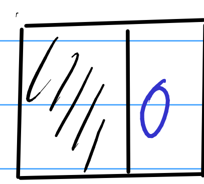
$$Q^T Q^{TT} = Q^T Q = \begin{pmatrix} - b_1 - \\ - b_2 - \\ \vdots \\ - b_n - \end{pmatrix} \begin{pmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_n \\ | & | & & | \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$$= I$$

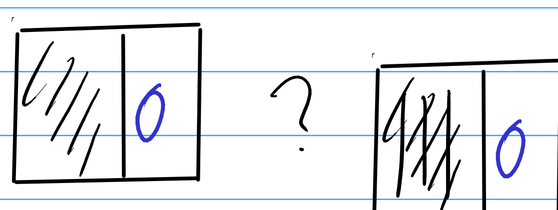
- $Q^{-1} = ?$ $\leadsto$ $Q^{-1} = Q^T$

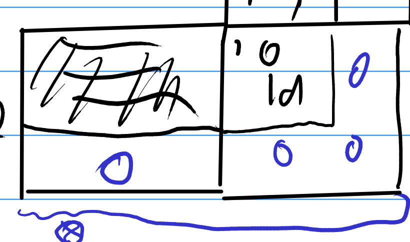
$$\left.\begin{aligned} AB &= I \\ BA &= I \end{aligned}\right\} \Rightarrow B = A^{-1}$$

- What if $b_1 \dots b_k$ don't span the whole space?

$$Q = \begin{pmatrix} | & | & & | & \\ b_1 & b_2 & \cdots & b_k & 0 \\ | & | & & | & \end{pmatrix} =$$ 

$P = QQ^T = \mathrm{Id}$?



$P^2 = Q\underbrace{Q^T Q}_{\otimes} Q^T = Q$  $Q^T = QQ^T = P$

P is an (orthogonal) projection

Any function f with $f(f(x)) = f(x)$ is a projection.

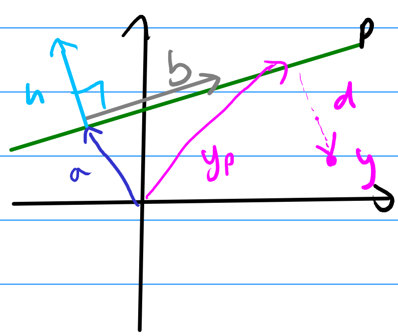
- Can also describe hyperplane using an inner product.

$n=2 \to$ line (1D)
$n=3 \to$ plane (2D)
$n=4 \to \ldots$ (3D)

dir. vect.

$P: \; x = a + \alpha b$

find n (normal) s.t. $b \perp n$



$$\langle x, n\rangle = \langle a + \alpha b, n\rangle$$
$$= \langle a, n\rangle + \alpha \cancel{\langle b, n\rangle}$$

For any $x \in P$: $\langle x, n\rangle = \langle a, n\rangle$ is another way of writing down an eqn. for the line (2D) P.

point-normal form

If $\|n\| = 1$, then $\langle y, n\rangle = \underbrace{\langle y_p, n\rangle}_{\langle a, n\rangle} + \langle d, n\rangle$

$$\langle d, n\rangle = \langle y, n\rangle - \underbrace{\langle y_p, n\rangle}_{\langle a, n\rangle}$$

$$\|d\| = \|\underbrace{\langle d, n\rangle n}_{d}\| = \left|\langle y, n\rangle - \langle a, n\rangle\right|$$