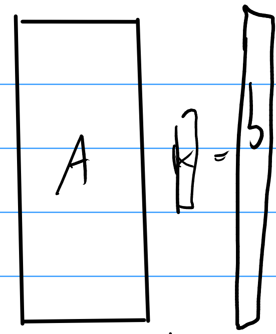
$Ax = b$



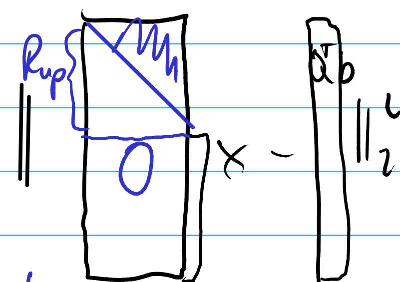
$A = QR \leftarrow$ complete $\leftarrow$ Q square $\rightarrow$ orthogonal

$$\|\underbrace{\overbrace{Ax-b}^{\text{residual}}}_{r}\|_2^2 = \|QRx-b\|_2^2$$

$$= \|Q^T(QRx-b)\|_2^2$$

$$= \|Rx - Q^Tb\|_2^2$$

$=$



$Q^TQ = I$

$\|Q^Ty\|_2^2 = \|y\|_2^2$

set x solve this

$$R_{up}x = (Q^Tb)_{up} \rightarrow (Rx - Q^Tb)_{up} = 0$$

$$(Rx - Q^Tb)_{lower} \neq 0$$

$$\|Ax-b\| = \|Rx - Q^Tb\|_2 = \sqrt{0^2+0^2+\cdots+0^2+\underbrace{\tilde{r}_k^2 + \cdots + \tilde{r}_n^2}_{\text{nonzero } Rx-Q^Tb}}$$

$$= \|(\cancel{Rx} - Q^Tb)_{lower}\|_2$$

(because of lower)

$$= \|(-Q^Tb)_{lower}\|_2 = \|(Q^Tb)_{lower}\|_2$$

Transformed residual $\tilde{r} = Q^T r = Q^T(Ax-b)$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ \otimes \\ \otimes \\ \otimes \end{pmatrix}$$

$\otimes$ $\tilde{r} \perp$ col space R

[Side fact: Orthogonal matrices preserve dot product

$x \cdot y = x^T y = x^T \underbrace{\cancel{Q^T Q}}_{Id} y = (Qx)^T (Qy) = (Qx)\cdot(Qy)$

$\otimes \iff \tilde{r} \cdot$ columns of $R = 0$

$\iff Q\tilde{r} \cdot Q$ (columns of R) $= 0$

$\iff r \cdot$ (columns of A) $= 0$

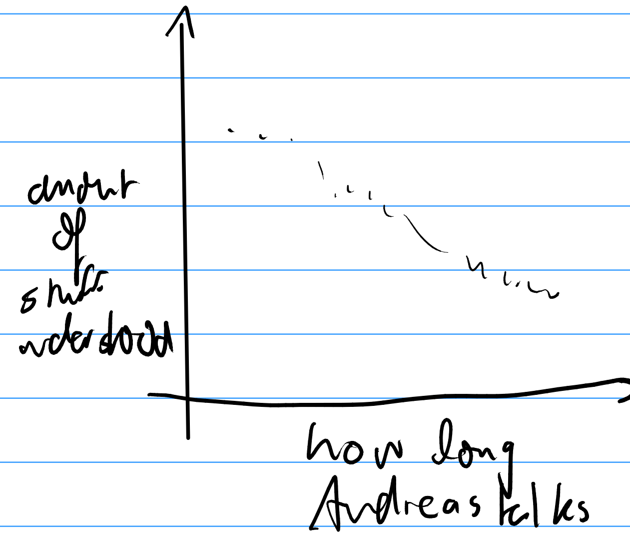
Fact: For the residual r of A least-squares problem $\min \|Ax-b\|_2$, the residual r

$r \perp$ column space of A.

Data fitting

Have: (x_i, y_i)

Have: $y = \underset{?}{a} + b\underset{?}{x}$



$$\begin{aligned} a + bx_1 &= y_1 \\ &\vdots \\ a + bx_n &= y_n \end{aligned} \quad \rightsquigarrow \quad \underbrace{\begin{pmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} a \\ b \end{pmatrix}}_{z} = \underbrace{\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}}_{c}$$

$$\| Az - c \|_2$$

$$\begin{aligned} a + bx_1 + cx_1^2 &= y_1 \\ &\vdots \\ a + bx_n + cx_n^2 &= y_n \end{aligned} \quad \rightsquigarrow \quad \begin{pmatrix} 1 & x_1 & x_1^2 \\ \vdots & & \vdots \\ 1 & x_n & x_n^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

New model: $y = a\cdot\sin(x) + b\cdot\cos(x)$

$$a\cdot\sin(x_1) + b\cdot\cos(x_1) = y_1$$
$$\vdots \qquad\qquad \vdots$$
$$a\cdot\sin(x_n) + b\cos(x_n) = y_n$$

$$\rightsquigarrow \begin{pmatrix} \sin(x_1) & \cos(x_1) \\ \vdots & \vdots \\ \sin(x_n) & \cos(x_n) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} =$$

Models solvable w/ least squares:

$$\underline{a_1}\cdot\varphi_1(x) + \underline{a_2}\cdot\varphi_2(x) + \dots + \underline{a_n}\varphi_n(x)$$

Can't solve:

$$\exp(\underline{a_1}x_1) + \sin(\underline{a_2}x)^{\underline{a_3}}$$

$$(Rx)_{upper} = (Q^T b)_{upper}$$

$$Q^T b$$

$$\boxed{\cancel{\begin{pmatrix} 3 \\ 0 \end{pmatrix}(x_1) = \begin{pmatrix} 3/\sqrt{2} \\ 2 \end{pmatrix}}} \qquad = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3/\sqrt{2} \\ ? \end{pmatrix}$$

$$3x_1 = 3/\sqrt{2} \qquad x_1 = 1/\sqrt{2}$$