Applications of the SVD

① Least squares problems

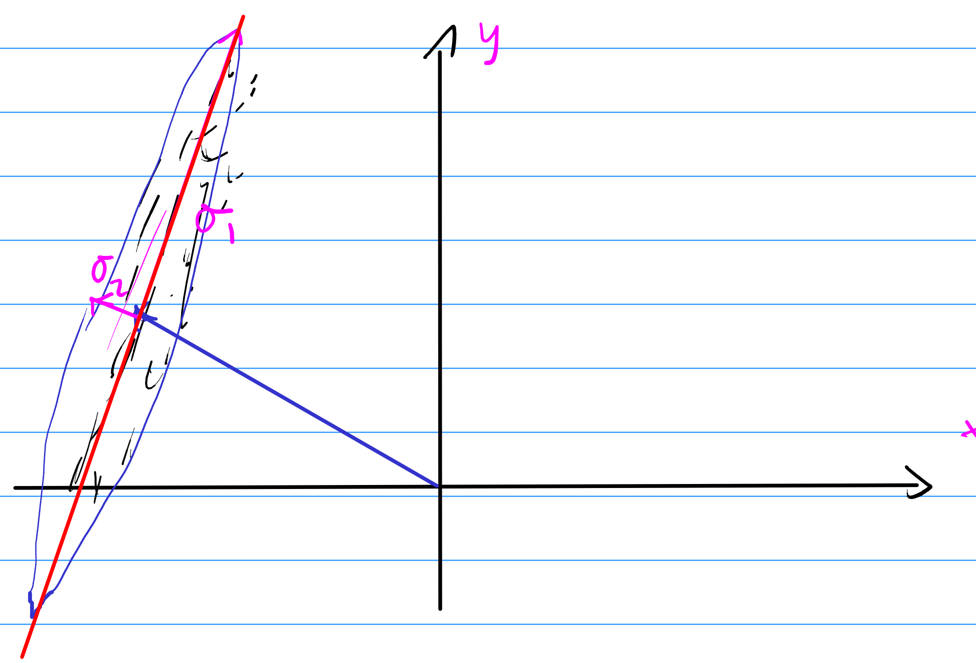
$x = A^+ b$ solves the least-squares problem

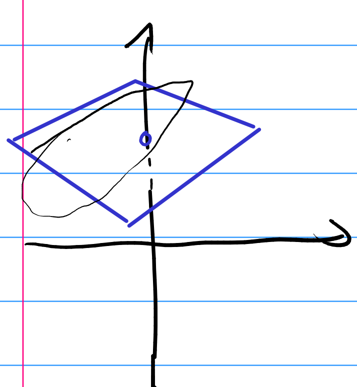
Choose x s.t. $\|Ax - b\|_2^2 \to \min!$ ⊗

If the solution to ⊗ is not unique,

then $A^+ b$ is the solution with the smallest 2-norm.

② PCA → HW





③ Computing $\|A\|_2$

$$A = \underset{U}{\begin{pmatrix} | & | & | & \cdots & | \end{pmatrix}} \underset{\Sigma}{\begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix}} \underset{V^T}{\begin{pmatrix} - \\ \vdots \\ - \end{pmatrix}}$$

$$\|A\|_2 = \sigma_1 = \|A v_1\|$$

Convention: $|\sigma_1| \geq |\sigma_2| \geq |\sigma_{\cdot} \ldots$

④ Computing $\kappa_2(A)$

$$\kappa_2(A) = \|A\|_2 \cdot \|A^{-1}\|_2$$

Assume A is invertible.

Solve $Ax = b$

$$A(x + \Delta x) = b + \Delta b$$

rel. error in b: $\frac{\|\Delta b\|}{\|b\|}$

$$\frac{\|\Delta x\|}{\|x\|} \leq \kappa(A) \cdot \frac{\|\Delta b\|}{\|b\|}$$

$$A = \underset{U}{\begin{pmatrix} | & | & | & \cdots & | \end{pmatrix}} \underset{\Sigma}{\begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix}} \underset{V^T}{\begin{pmatrix} - \\ \vdots \\ - \end{pmatrix}}$$

$$\kappa_2(A) = \sigma_1 \cdot \|A^{-1}\|_2 \qquad A^{-1} = V \Sigma^{-1} U^T$$

$$= \frac{\sigma_1}{\sigma_n}$$

← Formula for $\kappa_2(A)$ even if A is not invertible smallest

$$A^{-1} = V\Sigma^{-1}U^T = V \begin{pmatrix} 1/\sigma_1 & & \\ & \ddots & \\ & & 1/\sigma_n \end{pmatrix} U^T$$

biggest

$$\|A^{-1}\| = \frac{1}{\sigma_n}$$

5 Low-rank approximation

$$A = \underset{U}{\begin{pmatrix} | & | & | & | & | \\ | & | & | & | & | \end{pmatrix}} \underbrace{\underset{\Sigma}{\begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix}} \underset{V^T}{\begin{pmatrix} — \\ — \\ — \\ — \end{pmatrix}}}$$

$$\begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix} \begin{pmatrix} — v_1 — \\ \vdots \\ — v_n — \end{pmatrix} = \begin{pmatrix} — \sigma_1 v_1 — \\ — \sigma_2 v_2 — \\ \vdots \\ — \sigma_n v_n — \end{pmatrix}$$

$$\Sigma V^T = \begin{pmatrix} — \sigma_1 v_1 — \\ — \sigma_2 v_2 — \\ \vdots \\ — \sigma_n v_n — \end{pmatrix}$$

$$\begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix} \begin{pmatrix} — \sigma_1 v_1 — \\ \vdots \\ — \sigma_n v_n — \end{pmatrix} = \underbrace{u_1\sigma_1 v_1^T + u_2\sigma_2 v_2^T + \cdots\ u_n\sigma_n v_n^T}_{A}$$

The SVD yields an interpretation of A

as a sum of ~~outer products~~ rank-1 matrices. with decreasing norms

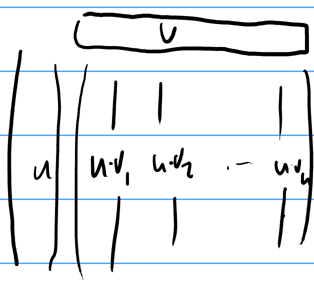
$$\begin{pmatrix} \alpha_1 v & \cdots & \alpha_n v \end{pmatrix}$$

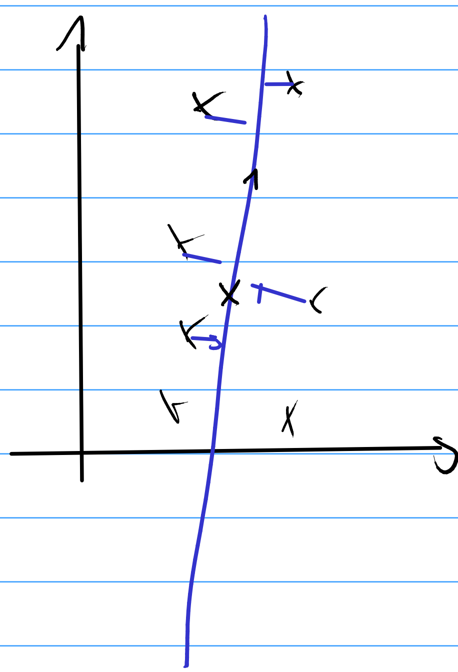
$$\|u_1 v_1^T\|_2 = 1$$

$$\|u_1 \sigma_1 v_1^T\|_2 = |\sigma_1|$$

Idea: Only use the most important rank-1 matrices (with big-ish singular vectors.)

$A_k = u_1\sigma_1 v_1^T + \cdots + u_k \sigma_k v_k^T$ is the rank-k matrix with minimal $\|A - A_k\|_F^2$.





$$Ax = \lambda x$$

$$y = 2x$$

$$\underline{Ay = A2x = \lambda 2x = \lambda y}$$

$Ax = \lambda x$ $\quad T^{-1}AT$ $\quad$ T invertible

$T^{-1}ATx \ldots ?$ $\quad y = T^{-1}x$

$$T^{-1}ATy = T^{-1}A\cancel{TT^{-1}}x = T^{-1}Ax = T^{-1}\lambda x$$
$$= \lambda(T^{-1}x) = \lambda y$$

$$(\lambda - 5 \cdot 1)^{-1} =$$