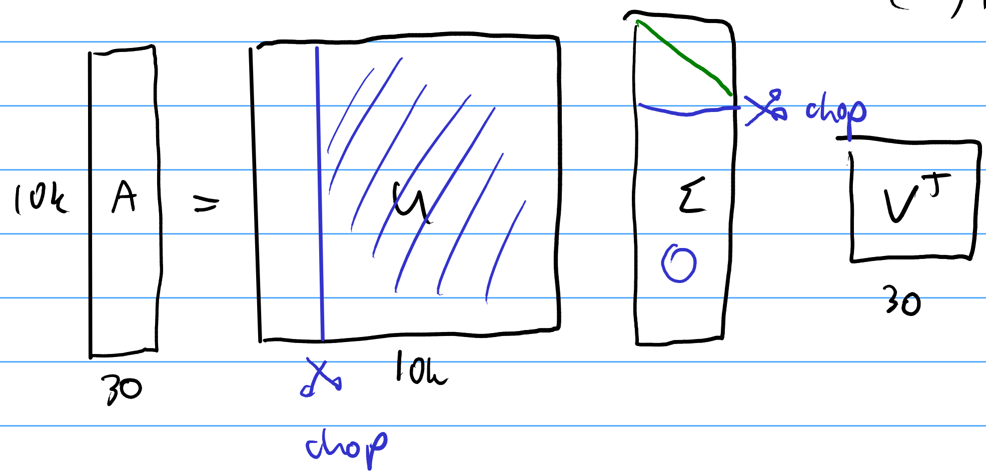
Thin SVD

$$(10^4)^2 10 = 10^9$$




```
np.linalg.svd(A, full_matrices=False)
```

Eigenvalues

$$Ax = \lambda x \qquad x \neq 0$$

$$A - \sigma I \qquad \rightsquigarrow \quad (A - \sigma I)\, x = (\lambda - \sigma)\, x$$

$$A^{-1} \qquad \rightsquigarrow \quad A^{-1} x = \frac{1}{\lambda} x$$

$$T^{-1} A T \qquad y := T^{-1} x$$

similarity

$$(T^{-1} A T)\, y = (T^{-1} A \underbrace{T)(T^{-1}}\, x)$$

$$= T^{-1} A x = T^{-1} \lambda x = \lambda T^{-1} x = \lambda y .$$

$$A = T^{-1} B T \quad \rightsquigarrow \quad A, B \text{ similar}$$

$$B = \underbrace{(T^{-1})^{-1}}\, A\, \underbrace{T^{-1}} \quad \Big| \; \underbrace{T^{-1}} \cdot$$

$$T^{-1} B = T^{-1} T A T^{-1} \quad \Big| \cdot T$$

$$T^{-1} B T = A$$

If A similar to a diagonal, then A is called diagonalizable.

$$T^{-1} \overset{D}{\begin{pmatrix} 5 & & \\ & 17 & \\ & & 2 \end{pmatrix}} T \quad \rightsquigarrow \quad \text{has eigenvalues } 5, 17, 2.$$

$$X = T^{-1}$$

$$X \begin{pmatrix} 5 & & \\ & 17 & \\ & & 2 \end{pmatrix} X^{-1}$$

$$\begin{pmatrix} 5 & & \\ & 17 & \\ & & 2 \end{pmatrix} \underbrace{\overset{e_1}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}}_{x} = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} = \underbrace{5}_{\lambda} \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{x}$$

$$\begin{pmatrix} 5 & & \\ & 17 & \\ & & 2 \end{pmatrix} \underset{e_2}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} = 17 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$De_1 = \lambda_1 e_1 \rightsquigarrow T^{-1}DT\,(T^{-1}e_1) = \lambda_1 (T^{-1}e_1)$$

Columns of X are eigenvectors.

<u>Are there always as many eigenvectors as eigenvalues?</u>

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ Eigenvalues: $1, 1$

$$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y \end{pmatrix} = 1 \cdot \begin{pmatrix} x \\ y \end{pmatrix} \rightsquigarrow \begin{matrix} x+y = x \rightsquigarrow y = 0 \\ y = y \end{matrix}$$

$$\begin{pmatrix} x \\ 0 \end{pmatrix} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

"algebraic" multiplicity of eigenvalue λ $\geq$ "geometric" multiplicity of eigenval λ

# of copies of <u>eigenvalue</u>

eigenvectors (l.i.) for λ

$>$ $\leftarrow$ matrix "defective"

XDX^{-1}

diagonalizable matrices are not defective

Tool Schur form

similarity transform

$A = Q U Q^T$ ← always exists

where
- U is upper
- Q is orthogonal $\rightsquigarrow Q^{-1} = Q^T$

Fact:
- eigenvalues of A are the entries on diagonal of U
- $Ux = \lambda x$? ← can we find eigenvectors?

$\rightsquigarrow Ux - \lambda x = 0$

upper

$\rightsquigarrow (U - \lambda I)x = 0 \rightsquigarrow x \in N(U - \lambda I)$

$A Qx = Q U Q^T Qx = QUx = Q\lambda x = \lambda Qx$

$Ax_1 = \lambda_1 x_1$ . . . $Ax_n = \lambda_n x_n$ x_n li!

$|\lambda_1| \geq$ . . . $|\lambda_n|$

$x = \alpha_1 x_1 + \cdots + \alpha_n x_n$

$A^{20,000} x = \alpha_1 \lambda_1^{20,000} x_1 + \cdots + \alpha_n \lambda_n^{20,000} x_n$

$\|Ax\| = \sqrt{x^T A^T A x}$

"Rayleigh quotient" $\frac{x^T A x}{x^T x} = \frac{x^T \lambda x}{x^T x} = \lambda$ $Ax = \lambda x$

$Ax = \lambda x$

$\frac{A^{20,000} x}{\lambda_1^{20,000} x} = \alpha_1 x + \alpha_2 \frac{\lambda_2}{\lambda_1^{20,000}} x_2 + \cdots$

A^{-1}

$(A - \sigma I)^{-1}$