Interpolation with a monomial basis

Space we're spanning is "polynomials up to degree $n-1$"

$x^0 = 1, \; x^1, \ldots, x^{n-1}$

second

first monomial

Look for $\tilde{f}(x) = \sum_{i=1}^{n} \alpha_i \varphi_i(x)$

Interpolation "basis"

$\alpha_1 \varphi_1 + \ldots + \alpha_n \varphi_n$

$\begin{pmatrix} \vdots \end{pmatrix} \leftarrow$ vector $\qquad f \leftarrow$ also a vector.

functions are vectors too!

Space of polynomials up to degree $n-1$:

$\beta_0 \cdot 1 + \beta_1 \cdot x^1 + \beta_2 x^2 + \ldots + \beta_{n-1} x^{n-1}$: monomials span the space ✓

Monomials lin indep.?

$x^k = \gamma_0 \cdot 1 + \gamma_1 \cdot x^1 + \ldots + \gamma_{k-1} x^{k-1} + \ldots + \gamma_{k+1} x^{k+1} \ldots + \gamma_{n-1} x^{n-1}$

(cannot find such γ_i)

$\Rightarrow$ monomials are linearly indep. ✓

$\Rightarrow$ monomials are a basis.

$\rightarrow$ Key assumption in in Interpolation:

functions = # points

$\Rightarrow$ gen. Vandermonde is square gen. Vdm. $\rightarrow$ interp. points (x_i) "nodes"

"generalized Vdm" "Vdm."

$$V = \begin{pmatrix} \varphi_1(x_1) & \cdots & \varphi_n(x_1) \\ \vdots & & \vdots \\ \varphi_1(x_n) & \cdots & \varphi_n(x_n) \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & \cdots & x_1^{n-1} \\ \vdots & & \vdots \\ 1 & \cdots & x_n^{n-1} \end{pmatrix}$$

for $\varphi_i(x) = x^{i-1}$

$$V \begin{pmatrix} a_0 \\ \vdots \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} f(x_1) \\ \vdots \\ f(x_n) \end{pmatrix}$$

Interp. error: $f \rightarrow$ "true function"
$\tilde{f} \rightarrow$ "interpolant"

$$\max_{x \in [0,n]} |f(x) - \tilde{f}(x)| \le C \cdot h^n$$

$$= C \cdot h^{\underbrace{(\#\text{points} - 1)}_{\text{max degree}} + 1}$$

does not depend, dep. on f, n

will start holding as soon as h is "small enough"

works only if function is smooth

Full QR: