

How wrong is my result?

I not a very precise q.

Model:

computed value = true value + error

$$\vec{x} = \vec{x}_0 + \vec{e}$$

Two error measures:

- absolute error $\|\vec{e}\|$
- relative error $\frac{\|\vec{e}\|}{\|\vec{x}_0\|}$

A more precise question: abs./rel. condition #

By what Factor does my calculation amplify the (abs./relative) error when I apply my computation?

"not much": well-conditioned

"a lot": poorly conditioned

Condition number of solving a lin. system:

$$A \overset{\text{output}}{x_0} = \overset{\text{input}}{b_0}$$

$$A(x_0 + \Delta x) = (b_0 + \Delta b)$$

$$\text{cond. \#} = \frac{\|\Delta x\|}{\|x_0\|} / \frac{\|\Delta b\|}{\|b_0\|}$$

$$= \frac{\|\Delta x\|}{\|x_0\|} \cdot \frac{\|b_0\|}{\|\Delta b\|}$$

$$= \frac{\|b_0\|}{\|x_0\|} \cdot \frac{\|\Delta x\|}{\|\Delta b\|}$$

$$= \frac{\|A x_0\|}{\|x_0\|} \cdot \frac{\|\Delta x\|}{\|\Delta b\|}$$

$$\leq \frac{\|A\| \cancel{\|x_0\|}}{\cancel{\|x_0\|}} \cdot \frac{\|A^{-1} \Delta b\|}{\|\Delta b\|} \quad \begin{array}{l} A \Delta x = \Delta b \\ \Delta x = A^{-1} \Delta b \end{array}$$

$$\leq \|A\| \cdot \frac{\|A^{-1}\| \cancel{\|\Delta b\|}}{\cancel{\|\Delta b\|}}$$

$$\leq \|A\| \cdot \|A^{-1}\|$$

↑
condition number of this matrix

$$\frac{\|Ax\|}{\|x\|} \leq \|A\|$$

$$\downarrow \\ \|Ax\| \leq \|A\| \|x\|$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

$$\|A\|_2 = 3$$

$$\|A^{-1}\|_2 = 1$$

$$A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$$

$$\kappa_2(A) = 3.$$