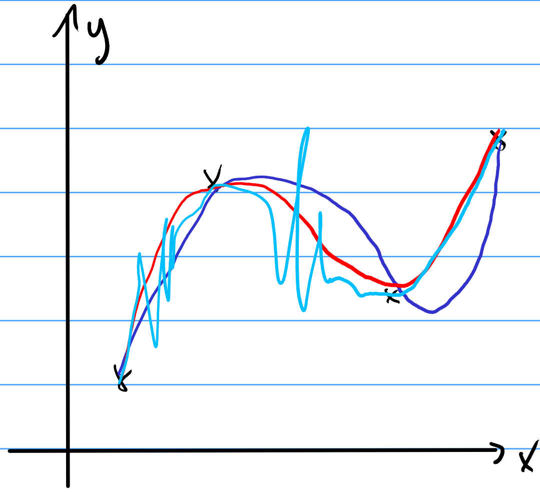
Interpolation



x	y
x_1	y_1
x_2	y_2
x_3	y_3
x_4	y_4

Goal: Find function that obeys $f(x_i)=y_i$

Seeking: $\alpha_1 \dots \alpha_m$

$$f(x)=\alpha_1\varphi_1(x)+\dots+\alpha_m\varphi_m(x)$$ the "interpolant"

Differences from LSQ data fitting:

- Hit function values exactly
- Number of coefficients matches # of data points

$$f(x_1) = \alpha_1 \varphi_1(x_1) + \cdots + \alpha_m \varphi_m(x_1) = y_1$$

$$\vdots$$

$$f(x_m) = \alpha_1 \varphi_1(x_m) + \cdots + \alpha_m \varphi_m(x_m) = y_m$$

$$\hookrightarrow \underbrace{\begin{pmatrix} \varphi_1(x_1) & \cdots & \varphi_m(x_1) \\ \vdots & & \vdots \\ \varphi_1(x_m) & \cdots & \varphi_m(x_m) \end{pmatrix}}_{} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$$

generalized Vandermonde matrix

$$V \begin{pmatrix} \text{basis} \\ \text{coeffs} \end{pmatrix} = \begin{pmatrix} \text{point} \\ \text{values} \end{pmatrix}$$

Example

m data points

Monomials: $x^0, x^1, \dots, x^{m-1}$

$\varphi_0 \quad \varphi_1 \quad \varphi_{m-1}$

$$\begin{array}{r c}
 & \begin{array}{cccc} \varphi_1 & & & \varphi_m \end{array} \\
\text{point } x_1 \rightarrow & \\
\text{point } x_m \rightarrow &
\end{array}
\qquad
V = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{m-1} \\ 1 & x_2 & x_2^2 & \cdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_m & x_m^2 & \cdots & x_m^{m-1} \end{bmatrix}$$

$\leftarrow$ "un-generalized" Vandermonde matrix

Example:

x_i	y_i
0	7
2	4
3	4

$$V = \begin{array}{c} \begin{array}{ccc} 1 & x & x^2 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 2^2 \\ 1 & 3 & 3^2 \end{pmatrix} \end{array} \qquad V\vec{\alpha} = \begin{pmatrix} 7 \\ 4 \\ 4 \end{pmatrix}$$

Have; $f(x) = \alpha_0 \cdot 1 + \alpha_1 \cdot x + \alpha_2 \cdot x^2$

Take derivative: $f'(x) = \alpha_0 \cdot 0 + \alpha_1 \cdot 1 + \alpha_2 (2x)$

$$V' = \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2x \end{array} \\ \begin{pmatrix} 0 & 1 & 2\cdot 0 \\ 0 & 1 & 2\cdot 2 \\ 0 & 1 & 2\cdot 3 \end{pmatrix} \end{array}$$

$$\vec{\alpha} = V^{-1} \begin{pmatrix} 7 \\ 4 \\ 4 \end{pmatrix}$$

$$V'\alpha = \begin{pmatrix} f'(0) \\ f'(2) \\ f'(3) \end{pmatrix} \quad \rightsquigarrow \quad \begin{pmatrix} f'(0) \\ f'(2) \\ f'(3) \end{pmatrix} = V' V^{-1} \begin{pmatrix} f(0) \\ f(2) \\ f(3) \end{pmatrix}$$

$V(\text{coeff}) = (\text{point values})$