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Optimization

Let's try to weaken the requirement $\vec{f}(\vec{x}) = \vec{0}$. $(f: \mathbb{R}^n \to \mathbb{R}^n)$

Idea: minimize $\|\vec{f}(\vec{x})\|_2$

Create a problem statement for "optimization".

$g: \mathbb{R}^n \to \mathbb{R}$ $g(\vec{x}) = \|\vec{f}(\vec{x})\|_2$

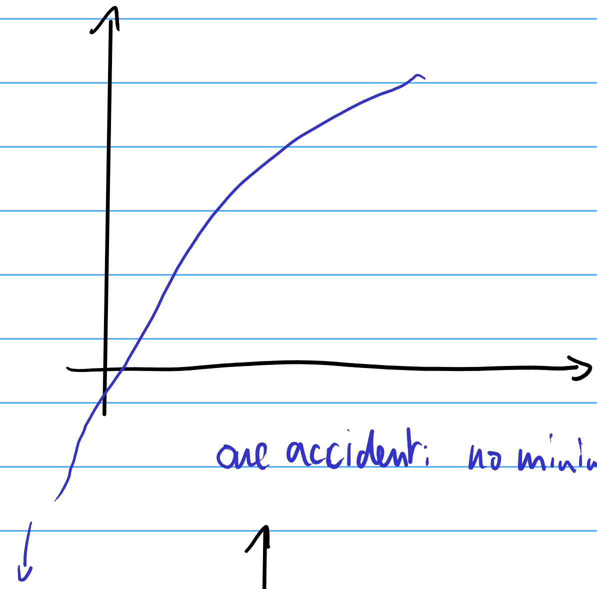
Find $\vec{x}$ where $g(\vec{x})$ assumes the smallest possible value

What if I'm interested in the largest possible value of a function g instead?

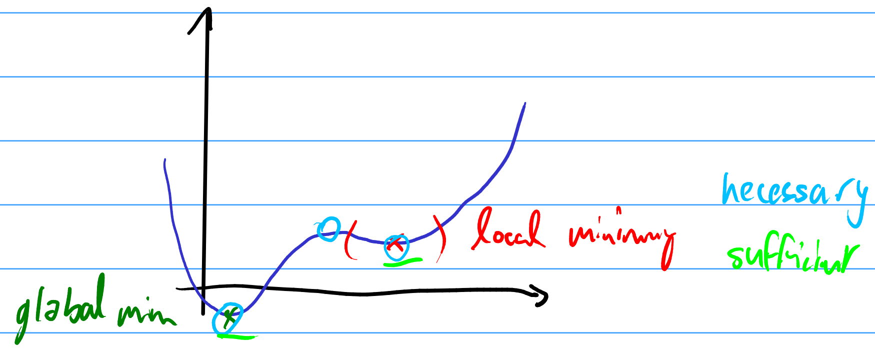
$\tilde{g}(x) = -g(x)$

$\vec{x}$ minimizes $\tilde{g}$ $\Leftrightarrow$ $\vec{x}$ maximizes g

What could go wrong?



one accident: no minimum



How can we tell if we've got a (local) minimum in 1D? Remember calculus!

necessary $\qquad g'(x) = 0$

sufficient $\qquad g'(x) = 0$ and $g''(x) > 0$

And in n dimensions?

necessary $\qquad \nabla g(x) = 0 \qquad \nabla g(x) = \begin{pmatrix} \frac{\partial g}{\partial x_1} \\ \vdots \\ \frac{\partial g}{\partial x_n} \end{pmatrix}$

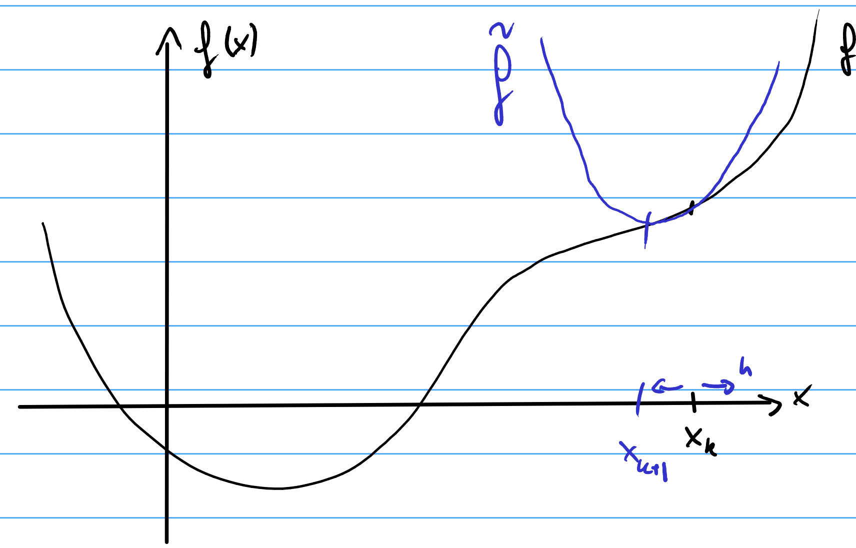
sufficient: $\qquad \nabla g(x) = 0$ and $H_g(x)$ positive def.

symmetric

$$H_g = \begin{pmatrix} \frac{\partial^2 g}{\partial x_1^2} & \cdots & \frac{\partial^2}{\partial x_1 x_n} g \\ & & \\ \frac{\partial^2}{\partial x_n x_1} g & & \frac{\partial^2}{\partial x_n^2} g \end{pmatrix}$$ Hessian

Let's steal the idea from Newton's method for equation solving.

Build a simple version of f and minimize that. Let's try in 1D first.



Does a linear approximation (a line) help at all? no.

from Taylor's theorem

$$\tilde{f}(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{2}$$

Now minimize that.

$$0 = \frac{\partial}{\partial h}\tilde{f} = f'(x) + f''(x)h$$

Newton's method (for opt.)

$$\rightsquigarrow \quad f''(x)h = -f'(x) \quad \rightsquigarrow \quad x_{k+1} = x_k - \frac{f'(x)}{f''(x)}$$

$$h = -\frac{f'(x)}{f''(x)}$$

$\rightsquigarrow$ equivalent to Newton for the eqn $f'(x) = 0$

Does that look at all familiar?