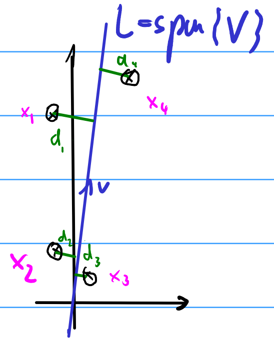
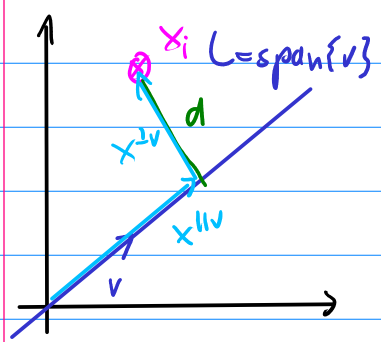


<u>Goal:</u> Find v with $\|v\|_2 = 1$ such that

$$\sum_{i=1}^{n} \operatorname{dist}(x_i, L)^2 = \sum_{i=1}^{n} \|d_i\|^2 \quad \text{is minimized.}$$

$\|\cdot\| = \|\cdot\|_2 \qquad (x,v) = x \cdot v$



$$x_i = x_i^{\perp v} + x_i^{\| v}$$

$$= x_i^{\perp v} + (x_i, v)v$$

$$\|\cdot\| \quad \|x_i\|^2 = \|x_i^{\perp v} + (x_i,v)v\|^2$$

$$\overset{\text{Pyth}}{=} \|x_i^{\perp v}\|^2 + \|(x_i,v)v\|^2$$

$$= \|x_i^{\perp v}\|^2 + (x_i,v)^2 \cancel{\|v\|^2}$$

$$\|d_i\|^2 = \|x_i\|^2 - (x_i,v)^2$$

$$\sum_i \|d_i\|^2 = \sum_{i=1} \|x_i\|^2 - (x_i,v)^2$$

$$\min \sum_i \|d_i\|^2 = \sum_{i=1}^{n}\left(\|x_i\|^2 - (x_i, v)^2\right)$$

$$= \|X\|_F^2 - \underbrace{\|X^T v\|^2}_{\max}$$

$$= \|X\|_F^2 - \|X\|_2^2$$

$$X = \begin{pmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{pmatrix}$$

$$X^T v = \begin{pmatrix} - x_1 - \\ \vdots \\ - x_n - \end{pmatrix} \begin{pmatrix} | \\ v \\ | \end{pmatrix} = \begin{pmatrix} x_1 \cdot v \\ \vdots \\ x_n \cdot v \end{pmatrix}$$

$$\max \|Av\| = \|A\| \quad \text{for } \|v\|=1$$

Find vector v_1 that maximizes $\|X^T v\|_2$ w/ $\|v_1\|=1$

→ first right singular vector

Call $\|X^T v_1\| = \sigma_1$ first singular value.

Find vector v_2 that maximizes $\|X^T v_2\|_2$ w/ $\|v_2\|=1$, $v_2 \perp v_1$.

Call $\|X^T v_2\| = \sigma_2$ second singular value.

$$V = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \qquad \Sigma = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix}$$

$$X = U \Sigma V^T$$

$$XV = U \cdot \Sigma$$

$$XV\Sigma^{-1} = U$$