

② Linear functions and matrices

Definition: A linear function f is a function $f: V \rightarrow W$ between two vector spaces V and W that respects the vector space operations:

- $f(x+y) = f(x) + f(y)$
- $f(\alpha x) = \alpha f(x)$

Which of these is linear as $f: \mathbb{R} \rightarrow \mathbb{R}$?

- Q [
- $f(x) = \sin x \quad \rightarrow \text{Try: ...}$
 - $f(x) = x^2$
 - $f(x) = a x + b$
 - $f(x) = a x$

Super-restrictive! Most functions are not linear.

- Q [
- What's $f(0)$ for f linear?

②.1 Linear functions and coordinates

If I have generators that span V : $v_1, v_2, v_3, \dots, v_m \in V$
 W : $w_1, w_2, w_3, \dots, w_n \in W$

then I can write down a linear function as a matrix.
Why?

Suppose

$$x = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_m v_m$$

then

$$f(x) = f(\alpha_1 v_1 + \dots + \alpha_m v_m)$$

$$= \alpha_1 f(v_1) + \dots + \alpha_m f(v_m)$$

↑
some combination
of w_1, \dots, w_n

↑
another combination
of w_1, \dots, w_n

$$\beta_{11} w_1 + \dots + \beta_{n1} w_n$$

$$\beta_{1m} w_1 + \dots + \beta_{nm} w_n$$

⋮

$$\begin{pmatrix} \beta_{11} & \dots & \beta_{1m} \\ \vdots & \ddots & \vdots \\ \beta_{n1} & \dots & \beta_{nm} \end{pmatrix} \begin{matrix} \xrightarrow{\alpha_1} \\ \xrightarrow{\alpha_m} \end{matrix} \begin{matrix} \gamma_1 \\ \gamma_n \end{matrix}$$

$$\approx f(x) = \gamma_1 w_1 + \gamma_2 w_2 + \dots + \gamma_n w_n$$

○ [So what does a matrix mean again?

Shorthand:

$$\begin{pmatrix} \beta_{11} & \dots & \beta_{1m} \\ \vdots & \ddots & \vdots \\ \beta_{n1} & \dots & \beta_{nm} \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix} = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix}$$

- Q [
- Coordinates of image of second gen. vector x_2 ?
 - What does an upper triangular matrix mean?
 - Mat-vec as indexy expression? einsum()?

lec 6

- discuss pre-quiz
- what's with the \mathbb{R}^T ?
- access to py examples?
- less abstract description of $f: V \rightarrow W$
- $f(v_3) \rightsquigarrow$ why not $(0, 0, \alpha_3)$?
- get γ from the matrix - where does w come from?

2.2 Chaining linear functions

$$\begin{array}{ccc} f: V \rightarrow W & g: W \rightarrow X & \text{both linear} \\ \downarrow & \searrow & \\ \text{represented as } f(x) = Ax & \text{represented as } g(x) = Bx & \end{array}$$

$$g(f(x)) = B \cdot (Ax)$$

Want a representation for $g \circ f \rightarrow g \circ f(x) = B(Ax) \stackrel{!}{=} (BA)x$

Where BA is a new matrix with

$$(BA)_{ij} = \sum_k B_{ik} A_{kj} \quad \leftarrow \text{matrix multiplication}$$

What if I want to chain three or more linear functions?

$$h: X \rightarrow Y \quad \leftarrow \text{represented as } h(x) = Cx$$

$$\begin{aligned} h(g(f(x))) &= C \cdot (B \cdot (Ax)) \stackrel{!}{=} (CB) \cdot (Ax) \\ &\stackrel{?}{=} C \cdot ((BA)x) \end{aligned}$$

Matrix multiplication is "associative" \rightarrow does not matter

$$\begin{array}{c} \uparrow \\ C(BA) = (CB)A = CBA \end{array}$$

2.3 What can matrices do?

- Geometry rotation Demo
- Represent graphs and walks Demo
- Blur on image Demo

lec 7

reminder: hw1 due in a week

numpy: what if axes are in wrong order? (einsum, broadcast) O expensive operation?

discuss quiz

Qs: best way to relearn linear algebra?

got responses first time?

standard basis = preferred linear indep. set?

hw due on cal

codes on hw1 in PDF; only for the sparse matrix and the planes

confused on hw1 p1? \rightarrow office hours / maybe extra mat

How do you test for lin. indep.?

2.4 Going backwards: Solving and Inverting

If a linear $f(x)$ applies some interesting operation, it would be useful to be able to undo this transformation, generically. (i.e. without thinking! All you'd need is the matrix.)

For an upper triangular matrix: no problem.

$$\begin{pmatrix} x & x & x & x \\ & x & x & x \\ & & x & x \\ & & & x \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

Can you code it?
Demo Backsubst.

General case of coefficient finding: more problematic

$$\begin{array}{ccc|c} \alpha_1 & \alpha_2 & \alpha_3 & \\ \hline 1 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 12 \end{array}$$

← want ← have

$$(\alpha_i) = (7 \ 0 \ 0) ?$$

$$(\alpha_i) = (0 \ 7 \ 12) ?$$

Q [What's the correct answer to this? Does it fit into a numpy array?

Define linear indep. to sidestep this problem.