

③ Basis and dimension

Suppose we have a vector space $V \subseteq \mathbb{R}^4$

with generators $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$.

o [Anything wrong with that?

o [How can we tell if there's a vector too many?

If we can get one as a linear combination of the others.

→ rearrange

Definition: A set of vectors v_1, \dots, v_n is called linearly dependent if and only iff

$$0 = \alpha_1 v_1 + \dots + \alpha_n v_n \quad \text{with at least one } \alpha_i \neq 0.$$

Opposite: linearly independent ← usually good / desirable

o [In case above, multiple α_i had to be non zero to get 0.
Can it happen that exactly one $\alpha_i \neq 0$ already yields 0?

Definition: Linearly independent generators of a vector space V are called a basis of V .

o [Rephrase: v_1, v_2, \dots, v_n are a basis of $V \Leftrightarrow$ (1) ... and (2) ...

Facts:

- Subsets of linearly independent vectors are also linearly independent. (Why? By contradiction)

- Can always reduce a finite set of vectors to be linearly independent.

Q [~~What's the basis of the vector space $\{0\}$?~~ (in quiz)

o [Can we always grow a set of vectors into a basis by adding vectors one by one?

- Let V be a vector space,
 $S = \{v_1, v_2, \dots, v_m\}$ a set of generators of V (i.e. they span V)
and $B = \{b_1, b_2, \dots, b_n\}$ a set of linearly indep. vectors in V .

Then $m \geq n$.

(Klein book: "Morphing lemma" for smuggling B into S)

- Consequence: (How? Suppose you have two bases...)

All bases of a given vector space have the same size.

→ Dimension of the vector space: dim V

o [How many dimensions in \mathbb{R}^2 ? \mathbb{R}^3 ?
~~a 100×100 image?~~ a bread recipe?
(in quiz)

- What does $V \subseteq W$ mean for $\dim V$ vs $\dim W$?
- What if $V \subseteq W$ and $\dim V = \dim W$?

- Consequence:

In a finite-dimensional vector space, you can always 'grow' a basis.

<inclass-lin-dep-basis>

lec 7

norms

computationally, too

align quiz + prelec?

quiz

rem hw1 due

wik reading

hw1 p1 - will improve test

0 [So linear (in)dependence seems useful to know about.
How do we test for it?

↳ Def. said $\alpha_1 v_1 + \dots + \alpha_n v_n = 0$

$$\begin{pmatrix} | & | & & | \\ \alpha_1 & \alpha_2 & & \alpha_n \\ | & | & & | \\ v_1 & v_2 & & v_n \\ | & | & & | \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

0 [still don't know how to solve that,
but matrix POV provides another way
of thinking about the problem.

eg. (v_i) linearly indep \Leftrightarrow ...?

Definition: Nullspace of a linear function: $N(f) = \{x : f(x) = 0\}$
↑ sometimes "kernel" or "ker"
↑ a vector space (why?)

Also: Nullspace of a matrix: $N(A) = \{x : Ax = 0\}$

dim $N(f)$ ← sometimes "nullity"

Seems sensible to look at linear independence through matrix goggles.
 → same stuff, new words.

$$A = \begin{pmatrix} | & | & & | \\ \beta_1 & \beta_2 & \dots & \beta_m \\ | & | & & | \end{pmatrix} \rightsquigarrow \text{span}(\beta_1, \dots, \beta_m) = \text{column space}(A)$$

$$\dim \text{column space}(A) = \text{col rank}(A)$$

o $\left[\text{rank} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} = ? \right.$

o $\left[\text{relation to } m? \right.$

Could also say:

$$A = \begin{pmatrix} \text{---} \bar{\beta}_1 \text{---} \\ \text{---} \bar{\beta}_2 \text{---} \\ \vdots \\ \text{---} \bar{\beta}_n \text{---} \end{pmatrix} \rightsquigarrow \text{span}(\bar{\beta}_1, \dots, \bar{\beta}_n) = \text{row space}(A)$$

$$\dim \text{row space}(A) = \text{row rank}(A)$$

o $\left[\text{in example above?} \right.$

Massively surprising!

Facts: • $\text{col rank}(A) = \text{row rank}(A) = \underline{\text{rank}}(A)$

- o $\left[\begin{array}{l} \bullet \text{ What does this mean?} \\ \bullet \text{ Rank of an outer product?} \\ \bullet \text{ Rank of a sum of outer products?} \\ \bullet \text{ Revisit image compression demo} \\ \bullet \text{ Can you see the rank?} \\ \bullet \text{ So what can't I do in that demo?} \end{array} \right.$

Massive ly surprising!

Facts, cont'd:

- For a matrix with n columns:
 $\text{rank } A + \dim \mathcal{N}(A) = n$

- What does this mean?
- If a matrix has full rank, ...
- If a matrix has rank zero, ...
- Do these count dimensions in the same vector space? (input/output?)
- Suppose I have a matrix with a zero column. (rank/nullsp conseq.?)
What can't I do by adding a lin. dep. column vector?

lec 8

review rank/nullspace

review quiz

reminders:

hw1 due

mt1 in a week

IEF

Q's:
 → fewer views
 videos shorter & to point
 what's on the test?

matrix square for LU decomp?

overview of HW1 /c

~/plane-patch.py

more rank examples?

readings

Rank/Nullspace examples:

• $n \times n \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$ rank? $\dim N(A)$?

• $m \times n \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & & 0 \end{bmatrix}$ rank? $\dim N(A)$?

• $n \times n \begin{bmatrix} \oplus & \oplus & \oplus & \oplus & \oplus & \oplus \\ & \oplus & \oplus & \oplus & \oplus & \oplus \\ & & \oplus & \oplus & \oplus & \oplus \\ & & & \oplus & \oplus & \oplus \\ & & & & \oplus & \oplus \\ & & & & & \oplus \end{bmatrix}$ rank? $\dim N(A)$?

o [Why is $\text{rank}(U) = \text{rank}(A)$ in $A=LU$?

Does the permutation matrix change anything?

Properties of linear functions (in terms of rank, $\dim N(A)$)

$f: V \rightarrow W$ linear $\dim(V) = n$ $\dim(W) = m$ matrix A repr. f

• one-to-one?

$f(x) = f(y) \Rightarrow x = y$ \leftarrow Def 1-to-1

Linearity: $f(x) = f(y) \Rightarrow f(x-y) = 0$

iff $\dim N(A) = 0$, then $f(x-y) = 0 \Rightarrow x-y = 0$.

• onto?

$f(V) = W$ \leftarrow Def onto

$\Leftrightarrow \dim f(V) = \dim(W) \Leftrightarrow \text{rank}(A) = m$

• invertible?

must satisfy $\overbrace{\text{rank}(A) = m}^{\text{onto}}$, $\overbrace{\dim N(A) = 0}^{1\text{-to-1}}$

fact (above): $\underbrace{\text{rank}(A)}_m + \underbrace{\dim N(A)}_0 = n \Rightarrow m = n \Rightarrow A$ square

name for matrix representing $f^{-1}: A^{-1}$

Rank and linear independence, computationally

- If I draw two random vectors, what's the probability that they'll be linearly dependent?
- If I draw two random vectors, what's the probability they'll point the same way?

⚠ Keep this in mind - will be super useful later!

Computers don't represent numbers exactly. (in floating point)

Every FP number: 3.14159 777777
good digits junk

Model as: true answer + small "random" error

In a vector: 

o] What does this mean for a computational test of linear dependence?

Demo