④ Norms

Definition A norm is a function $\|\cdot\|$ (input) from a vector space V into the real numbers that satisfies:

- $\|x\| \geq 0$
- $\|x+y\| \leq \|x\| + \|y\|$
  (triangle inequality)
- $\|\alpha x\| = |\alpha| \, \|x\|$
- $\|x\| = 0 \Leftrightarrow x = 0$

Examples ("p-norms")

Let $p \geq 1$. Then we define for a vector x with coordinates $(x_1, \dots, x_n)$

$$\|x\|_p = \sqrt[p]{|x_1|^p + |x_2|^p + \dots + |x_n|^p}.$$

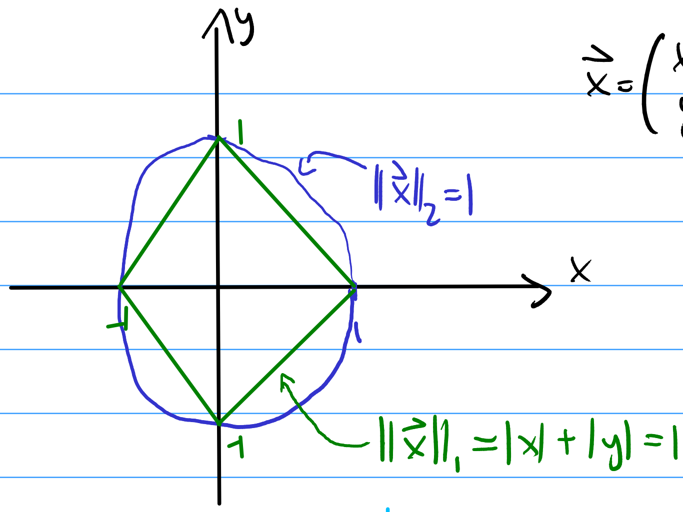
$$\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}.$$

$$\|x\|_1 = |x_1| + \dots + |x_n|$$

$$\|x\|_\infty \text{ “}=\text{” } \sqrt[\infty]{|x_1|^\infty + |x_2|^\infty + \dots + |x_n|^\infty}$$

$$= \max(|x_1|, |x_2|, \dots, |x_n|)$$

$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$



$\|x\|_\infty = 1$?

Definition The norm of a linear function f is defined as the "maximum stretch factor":

$$\|f\| = \max_{x \neq 0} \frac{\|f(x)\|}{\|x\|}$$

Comments:

- ⚠ This definition uses a (vector) norm. Using a different vector norm will change the function norm.
- This is the same as $\|f\| = \max_{\|x\|=1} \|f(x)\|$ (Why?)
- A matrix norm is defined the same way:

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

- Matrix norms often inherit the subscript p if they come from one of the p-norms:

$$\|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$