(8) Eigenvalues

$$A \in \mathbb{R}^{n \times n}$$

x is an eigenvector of A :

$x \neq 0$, there is a λ so that $Ax = \lambda x$

[eigenvectors unique?

No.

[must A be square?

Yes.

λ eigenvalue of A

$\Leftrightarrow$ $(A - \lambda I)x = 0$ has a solution $x \neq 0$

$\Leftrightarrow$ $N(A - \lambda I) \neq \{0\}$

$$\begin{pmatrix} 1 & x & x & x & x \\ & 2 & \ddots & & \vdots \\ & & 3 & \ddots & \\ & & & 4 & \\ & & & & 7 \end{pmatrix}$$

For ∇ matrices, eigenvalues appear on the diagonal.

Because eigenvalues satisfy a polynomial equation of degree n, and because there is no formula to solve such equations, num. methods for eigenvalues <u>must</u> be approximate!

What matrix operations do to eigenvalues

Suppose $Ax = \lambda x$.

- "Scaling" βA $\rightsquigarrow$ $(\beta A)x = \beta\lambda x$
- "Shift" $A - \sigma I$ $\rightsquigarrow$ $(A - \sigma I)x = Ax - \sigma x = (\lambda - \sigma)x$
- Power A^k $\rightsquigarrow$ $A^k x = \lambda^k x$
- Inverse A^{-1} $\rightsquigarrow$ $Ax = \lambda x \rightsquigarrow \frac{1}{\lambda}x = A^{-1}x$
- Similarity $T^{-1}AT$ $\rightsquigarrow$ $y := T^{-1}x$

$$\begin{aligned}(T^{-1}AT)\,y &= T^{-1}Ax \\ &= \lambda T^{-1}x \\ &= \lambda y\end{aligned}$$

$\rightsquigarrow$ Similarity transforms preserve eigenvalues ⚠

Will also say: A, B are similar if there exists a matrix T such that $B = T^{-1}AT$.

If A is similar to a diagonal matrix, it's called diagonalizable.

Let $X = (||||)$ be a matrix full of linearly independent eigenvectors. Then

$$AX = X \underbrace{\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}}_{D} \quad \rightsquigarrow \quad X^{-1}AX = D.$$

So that would do it (= diagonalize a matrix).

algebraic multiplicity	$\geq$	geometric multiplicity
(multiplicity of λ as a root of the characteristic polynomial)		(# of lin. indep. eigenvectors corresponding to λ)

Defective matrix has eigenvalue with

$$AM > GM.$$

Defective matrices are not diagonalizable.

Diagonalizable matrices are not defective.

Tool: Schur form

Every matrix is orthonormally similar to an upper triangular matrix.

$$A = QUQ^T$$

If we knew how to compute this, why would it be helpful for knowing eigenvalues? on the diagonal of U.

Do we have the eigenvectors, too? No, but easy to find.

Finding eigenvectors from Schur form (for eigenvalue λ)

① Find x with $Ux = \lambda x \iff x \in N(U - \lambda I)$

→ Null space finding, U is already upper tri.

② $y = Qx$ is an eigenvector of A:

$$Ay = QUQ^TQx = \dots = \lambda y$$

Power iteration

A diagonalizable — $n\times n$ — have $x_1,\dots,x_n$ lin. indep. eigenvectors with $\lambda_1,\dots,\lambda_n$ eigenvalues

$$|\lambda_1| \geq |\lambda_2| \dots \geq |\lambda_n|$$

$$x = \alpha_1 x_1 + \dots + \alpha_n x_n$$

$$Ax = \alpha_1 A x_1 + \dots + \alpha_n A x_n$$
$$= \alpha_1 \lambda_1 x_1 + \dots + \alpha_n \lambda_n x_n$$

$$A^{20,000} x = AA\cdot A \dots Ax$$

$$= \alpha_1 \lambda_1^{20,000} \cdot x_1 + \dots + \alpha_n \lambda_n^{20,000} x_n$$

$$\frac{A^{20,000} x}{\lambda_1^{20,000}} = \alpha_1 x_1 + \alpha_2 \underbrace{\frac{\lambda_2^{20,000}}{\lambda_1^{20,000}}}_{|\cdot|\leq 1} x_2 + \dots + \alpha_n \frac{\lambda_n^{20,000}}{\lambda_1^{20,000}} x_n$$

$$= \left(\frac{\lambda_2}{\lambda_1}\right)^{20k} \ll 1!$$

$$\left(\frac{\lambda_n}{\lambda_1}\right)^{20k}$$

Possible problems:

- starting vector has no component for λ_1 ← no problem because of rounding
- $\lambda_1 = \lambda_2$
- complex eigenvalues?

Demo: Power iteration and variants

Power it: $x_{k+1} = A x_k$

Normalized power it. $x_{k+1} = \frac{A x_k}{\|A x_k\|}$ ← fixes overflow issue

Inverse iteration: $x_{k+1} = \frac{A^{-1} x_k}{\| A^{-1} x_k \|}$ ← finds eigvec of smallest eigenvalue (by magnitude)

Inverse iteration w/ shift $x_{k+1} = \frac{(A - \sigma I)^{-1} x_k}{\| (A - \sigma I)^{-1} x_k \|}$ ← finds eigenvalues near σ

Rayleigh quotient iteration: set σ to the Rayleigh quotient in inv. it. w/ shift.

The Rayleigh quotient is an estimate of the eigenvalue for an approx. eigenvector x:

$$\frac{x \cdot Ax}{x \cdot x}$$ ← what if $Ax = \lambda x$ exactly?

lec19

exam Tue
practice exam out tomorrow
hw4, hw3 solutions out today
outline
recap: inverse iteration
discuss quiz

Q's: - diff btw normalized and un-normalized pwr it
- what is the point of pwr it if there is np.linalg?
- how come we can find the orig. eigval despite shift etc?
- how to get all eigvals?

Downside of all these methods? One eigenvector at a time.

Idea: Just iterate multiple vectors at a time.

→ "Simultaneous iteration"

0 Why doesn't that work? → all vectors converge to the eigenvector for λ_1 (the largest by magnitude)

Idea: Keep them linearly independent. Even better: keep them orthogonal!

Orthogonal iteration

$X_0 \in \mathbb{R}^{n \times p}$ $\quad$ $(p \leq n)$ arbitrary (ideally w/ full rank)

$Q_1 R_1 = X_0$

$X_1 = A Q_1$

$Q_2 R_2 = X_1$

$X_2 = A Q_2$

$\vdots$

Suppose this "converges" so that $X_{k+1} \approx X_k$.

$\rightsquigarrow \quad Q_{k+1} R_{k+1} = X_k \approx X_{k+1}$

$\rightsquigarrow \quad X_{k+1} = A Q_{k+1}$

$\rightsquigarrow \quad Q_{k+1} R_{k+1} = A Q_{k+1}$

$\rightsquigarrow \quad Q_{k+1} R_{k+1} Q_{k+1}^T = A \quad \leftarrow$ [? $\qquad$ Schur form!

"Professional" eigenvalue software uses the "QR algorithm" which is a variant of this.

<inclass-power-iteration>

lec 20

hw4 due date next Thu

outline

Q's

Computing the SVD

Assume we know the SVD $A = U \Sigma V^T$.

Then $A^TA = V \Sigma^T U^T U \Sigma V^T = \underbrace{V \Sigma^T \Sigma V^T}$

eigenvalues? $\sigma_1^2, \sigma_2^2, \dots, \sigma_n^2$

eigenvectors? $v_1, v_2, \dots, v_n$

Idea: To compute the SVD:

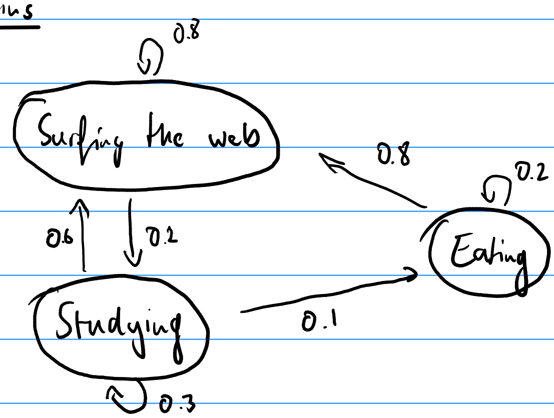
1. compute the eigenvalues $\lambda_1 \dots \lambda_n$ / eigenvectors of $v_1, \dots, v_n$ of A^TA.
   (e.g. Using orthogonal iteration)

2. Set $V = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$, $\Sigma = \begin{pmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{pmatrix}$

3. Find U from $A = U \Sigma V^T$.

Demo Computing the SVD

Markov chains



Important assumption: Only most recent state matters to determine the probabilities for the next state.
("Markov assumption/property")

Write transition probabilities into matrix:

$$
A = \begin{array}{c} \\ \\ \\ \\ \end{array}
\overbrace{\begin{array}{ccc} \text{surf} & \text{study} & \text{eat} \end{array}}^{\text{from state}}
$$

$$
A = \left(\begin{array}{ccc} .8 & .6 & .8 \\ .2 & .3 & 0 \\ 0 & .1 & .2 \end{array}\right)
\left.\begin{array}{l} \text{surf} \\ \text{study} \\ \text{eat} \end{array}\right]\text{to state}
$$

[Column sums = 1. Why?

State transitions modeled by matrix vector product.
Vector entries: Probability of being in i-th state.

$$A \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\text{surf}} = \begin{pmatrix} .8 \\ .2 \\ 0 \end{pmatrix} \qquad A \begin{pmatrix} .5 \\ .5 \\ 0 \end{pmatrix} = \begin{pmatrix} .4 + .3 \\ .1 + .15 \\ .05 \end{pmatrix}$$

If we observe all 50k UIUC students, what fraction of them are in each state?

Assume equilibrium: Previous state = next state.

$\rightsquigarrow Ax = \lambda x \quad \rightsquigarrow$ eigenvalue problem!

Demo: Find equilibrium distribution using power iteration