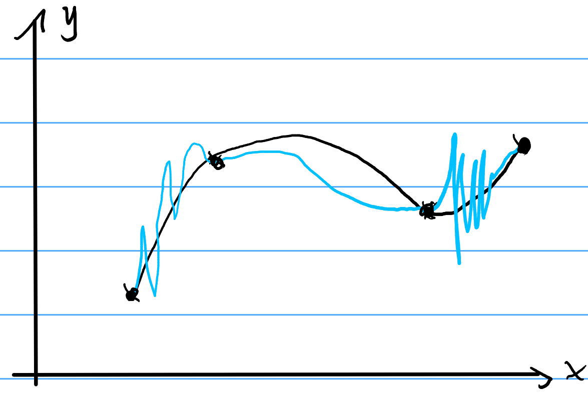
9 Interpolation and Function spaces



given

x	y
x_1	y_1
x_2	y_2
x_m	y_m

Goal: Find function through points

Really non-unique → need better specification of the problem

Seeking: $\alpha_1, \dots, \alpha_m$

in

linear

$$f(x) = \sum_{j=1}^{m} \alpha_j \varphi_j(x) \quad \text{s.t.} \quad f(x_i) = y_i \quad (i=1 \dots m)$$

$$\downarrow$$

$$f(x_i) = \sum_{j=1}^{m} \underbrace{\varphi_j(x_i)}_{V_{ij}} \alpha_j = (V\vec{\alpha})_i$$

generalized Vandermonde matrix

Interpolation: $V\vec{\alpha} = \vec{y}$

$$V \begin{pmatrix} \text{basis} \\ \text{coeffs} \end{pmatrix} = \begin{pmatrix} \text{node} \\ \text{values} \end{pmatrix}$$

Example: Interpolation with monomials

Monomial: $\underbrace{x^0}_{1}, x^1, x^2, x^3, x^4, x^5, \ldots$ [highest degree for m points? $m-1$

↳ Looking for (α_i) in $\alpha_0 x^0 + \alpha_1 x_1 + \alpha_2 x$

V for monomials:

$$V = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^4 \\ 1 & x_2 & & & \vdots \\ 1 & x_3 & & & \vdots \\ 1 & x_4 & & & \\ 1 & x_5 & & \cdots & x_5^4 \end{bmatrix}$$

(un-generalized) Vandermonde matrix

[Does this look at all familiar?

Yes, this is exactly what we've been doing in data fitting! Just now we're only using square matrices.

Example:

x_i	y_i
0	7
2	4
3	4

$$V = \begin{array}{c} \begin{matrix} 1 & x & x^2 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 2^2 \\ 1 & 3 & 3^2 \end{pmatrix} \end{array} \qquad V\vec{\alpha} = \underbrace{\begin{pmatrix} 7 \\ 4 \\ 4 \end{pmatrix}}_{\vec{b}}$$

$\rightarrow \vec{b} = V^{-1}\vec{\alpha}$

Have: $f(x) = \alpha_0 1 + \alpha_1 x + \alpha_2 x^2$

<u>Example of how an interpolant is helpful:</u> Take derivatives

Have: $f(x) = \alpha_0 \underbrace{1}_{\varphi_0(x)} + \alpha_1 \underbrace{x}_{\varphi_1(x)} + \alpha_2 \underbrace{x^2}_{\varphi_2(x)}$

Want: $f'(x) = \alpha_0 \underbrace{0}_{\varphi_0'(x)} + \alpha_1 \underbrace{1}_{\varphi_1'(x)} + \alpha_2 \underbrace{2x}_{\varphi_2'(x)}$

<u>Idea:</u> Build g. Vandermonde for the derivatives

$$V' = \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2x \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2\cdot 2 \\ 0 & 1 & 2\cdot 3 \end{pmatrix} \end{array} \rightsquigarrow V'\alpha = \begin{pmatrix} f'(0) \\ f'(2) \\ f'(3) \end{pmatrix}$$

$\rightsquigarrow$ $V'V^{-1}$ takes derivative of point values

coefficients $\vec{\alpha}$ are never computed!

Demo: Interpolation with generalized Vandermonde matrices

lec21

outline

review monomial interp.

review monomial interp. error

discuss quiz

Q's:
- functions are vectors!
- "monomial basis"
- how can we know the poly degree from # points?
- which is the best h?
- how is small h useful?
- Who depends on what in the error est.?
- interp. error discussion only true for smooth functions
- interp. error est. only for sufficiently small h

Interpolation and error

will resume list of bases in a bit

Demo: interpolation error part 1

Assumptions:
- f is "smooth" (has lots of derivatives)
- h "sufficiently small"

Fact: $|f(x) - \tilde{f}(x)| \le C \cdot h^{n+1}$ for all x

"true" function   interpolant

h: length of interval
n: highest polynomial degree in interpolant
C: an unknown constant (depends on f and n - but not on h!)

[What good is an estimate like this?

Helps predict errors.

[Why do shorter intervals (length h) help? function varies less on shorter interval

Suppose I have an interpolation error of E for some mesh spacing h using quadratic functions.

[What happens if I use $\frac{h}{2}$ instead of h?

$$\text{Error}(h) \approx C \cdot h^3$$

$$\text{Error}\left(\tfrac{h}{2}\right) \approx C \cdot \left(\tfrac{h}{2}\right)^3 = \tfrac{1}{8} \cdot C \cdot h^3 = \tfrac{1}{8} \cdot \text{Error}(h)$$

that's a concrete prediction!

Demo: interpolation error part 2

$\text{Error} \le C \cdot h^p$ ← "p^{th} order convergent"

[What is p here?

$n+1$

Some function bases

LEFT OUT

① Newton basis

Other bases for polynomials possible. One example:

$1, (x-x_1), (x-x_1)(x-x_2)$

zero at x_1: leaves value at x_1 alone

zero at x_1, x_2: leaves values at x_1 and x_2 alone

Shape of the generalized Vandermonde for Newton? lower triangular

② Monomial basis: $1, x, x^2, \dots, x^{n-1}$

Discussed above.

Demo: Monomial interpolation

Works fine up to $n=6$, after that conditioning starts to hurt.

→ Idea: Use multiple low-degree pieces!

lec22

Q's -->

review quiz review q

outline

pw/orth poly

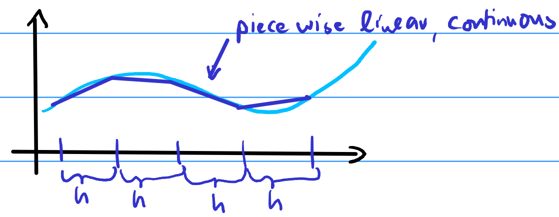
re-intro fourier

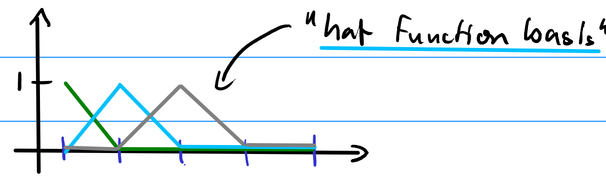
fourier apps

Q: What's an interval?

Why are we learning this?

③ Piecewise polynomials

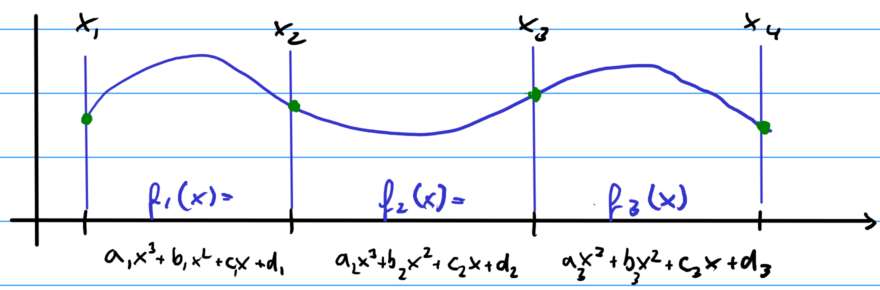




[Can you think of a basis?

[What does the generalized Vandermonde matrix look like for the hat function basis? Identity

Could also ask for piecewise polynomial with higher degree e.g. cubic ("splines")



• ← given points

→ 4 coefficients per panel to find: need to come up with conditions

→ more coefficients than data points

→ Vandermonde not super-helpful

→ HW5

[Again, why care about higher than n=1 degree?

Much smaller error for smooth functions.

What if I don't want to use piecewise interpolation?

[What is a general recipe for improving conditioning? Orthogonality

④ Orthogonal polynomials

[What does it mean for two functions to be orthogonal?

[Are functions vectors at all?

Need: inner product $\rightarrow$ $(f,g) := \int_0^1 f(x) g(x)\, dx$

Demo: Constructing Orthogonal polynomials

[What's the point of orthogonal polynomials?

Keep conditioning usable for higher n.

Choosing nodes for polynomial interpolation

Observation Discussed choice of basis

Also have choice of nodes

So far: only used equispaced nodes

Demo: Choice of interpolation nodes

Observation: Best if nodes cluster towards interval ends

Best nodes set on [-1,1]: "Chebyshev nodes"

$$x_k = \cos\left(\frac{2k-1}{2n}\pi\right) \qquad k=1\ldots n$$

5 Sines and Cosines ("Fourier Basis")

What's the basis? $\cancel{\sin(0x)}_{\to 0}$, $\cos(0x)$, $\sin(1x)$, $\cos(1x)$, $\sin(2x)$, $\cos(2x)$, ...

What to use as points? equispaced on $[0, 2\pi)$

$$2\pi\tfrac{0}{5},\ 2\pi\tfrac{1}{5},\ 2\pi\tfrac{2}{5},\ 2\pi\tfrac{3}{5},\ 2\pi\tfrac{4}{5},\ 2\pi\tfrac{5}{5}$$

no – $\cos(n\,2\pi) = \cos(0)$
$\sin(n\,2\pi) = \sin(0)$

Write down a Vandermonde matrix for up to n=2

$$\begin{matrix}
\cos(0\cdot 0) & \sin(1\cdot 0) & \cos(1\cdot 0) & \sin(2\cdot 0) & \cos(2\cdot 0)\\
\cos(0\,\frac{2\pi}{5}) & \sin(1\cdot\frac{2\pi}{5}) & \cos(1\cdot\frac{2\pi}{5}) & \sin(2\cdot\frac{2\pi}{5}) & \cos(2\cdot\frac{2\pi}{5})\\
\cos(0\,\frac{2\pi\cdot 2}{5}) & \sin(1\cdot\frac{2\pi\cdot 2}{5}) & \cos(1\cdot\frac{2\pi\cdot 2}{5}) & \sin(2\cdot\ \ ) & \cos(2\cdot\ \ )\\
\cos(0\,\frac{2\pi\cdot 3}{5}) & \sin(1\cdot\ \ ) & \cos(1\cdot\ \ ) & \sin(2\cdot\ \ ) & \cos(2\cdot\ \ )\\
\cos(0\,\frac{2\pi\cdot 4}{5}) & \sin(1\cdot\ \ ) & \cos(1\cdot\ \ ) & \sin(2\cdot\ \ ) & \cos(2\cdot\ \ )
\end{matrix}$$

Demo: Fourier Interpolation

Observations:

- works best for periodic functions
- generalized Vandermonde has orthogonal columns

Why care about the Fourier basis?

Demo: Audio synthesis with sines

Because it lets us think about frequency components

Demo: Audio experiments

Taking derivatives with linear algebra

$f(x_i) = \tilde{f}(x_i) \quad i=1\dots n$

Have: interpolant $\tilde{f}(x) = \alpha_1 \varphi_1(x) + \dots + \alpha_n \varphi_n(x)$

Want: derivative $\tilde{f}'(x) = \alpha_1 \varphi_1'(x) + \dots + \alpha_n \varphi_n'(x)$

Easy because interpolation basis (φ_i) is known!

Have: function values at nodes (x_i) $f(x_i)$

Want: values of derivative at nodes $f'(x_i)$ ← hard to get

$\tilde{f}'(x_i)$ ← easy to get!

How?

1. Compute coefficients $\vec{\alpha} = V^{-1}\vec{f}$ $\quad \vec{f} = \begin{pmatrix} f(x_1) \\ \vdots \\ f(x_n) \end{pmatrix}$

2. Build generalized Vandermonde with derivatives of basis

$$V' = \begin{pmatrix} \varphi_1'(x_1) & \cdots & \varphi_n'(x_1) \\ \vdots & & \vdots \\ \varphi_1'(x_n) & \cdots & \varphi_n'(x_n) \end{pmatrix}$$

3. Compute $V'\vec{\alpha} = \begin{pmatrix} \alpha_1 \varphi_1'(x_1) + \dots + \alpha_n \varphi_n'(x_1) \\ \vdots \\ \end{pmatrix} = \underbrace{\begin{pmatrix} \tilde{f}'(x_1) \\ \vdots \\ \tilde{f}'(x_n) \end{pmatrix}}_{\vec{\tilde{f}}'}$

Now all in one step: $\vec{\tilde{f}}' = \underbrace{V'V^{-1}}_{\text{matrix to apply a derivative (!)}}\vec{f}$

[What if you wanted coefficients of the derivative? $V^{-1}V'V^{-1}\vec{f}$

Demo: Taking derivatives with Vandermonde matrices

Observation: $|f' - \tilde{f}'| \le C \cdot h^n$ ← one less power than interpolation (where n is the highest poly degree)

Do we really need a whole matrix for equispaced points?

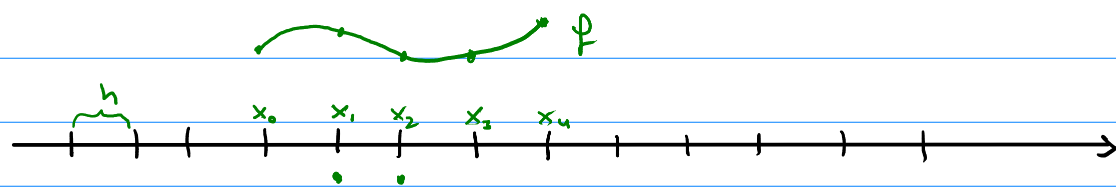
No! For equispaced data, one row of $V'V^{-1}$ should be enough.

(because taking a derivative is--in principle--no different there than at neighboring points)

Finite Differences

Idea: Reuse one of the 'middle' rows of $V'V^{-1}$ for many adjacent points:

$$V'V^{-1} = \begin{pmatrix} \otimes & \otimes & \otimes \\ -\frac{1}{2h} & 0 & \frac{1}{2h} \\ \otimes & \otimes & \otimes \end{pmatrix} \qquad V'V^{-1}\begin{pmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \end{pmatrix} = \begin{pmatrix} \otimes \\ -\frac{1}{2h} f(x_0) + \frac{1}{2h} f(x_2) \\ \otimes \end{pmatrix}$$



$$f'(x_1) = \frac{1}{2h} \cdot [f(x_2) - f(x_0)]$$

$$f'(x_2) = \frac{1}{2h} \cdot [f(x_3) - f(x_1)]$$

Demo: Finite differences

How accurate are these finite difference formulas?

$|f' - \tilde{f}'| \le C \cdot h^n$ ← as above! (It's the same thing after all.)

[How do I take two derivatives?

$$V' V^{-1} V' V^{-1} \vec{f}$$

Computing Integrals with Linear Algebra

Same idea as derivatives: Interpolate, then integrate.

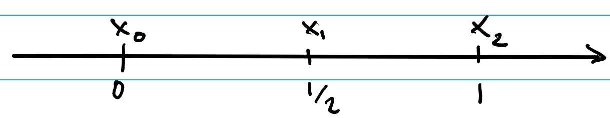
$f(x_i) = \tilde{f}(x_i) \quad i=1 \ldots n$

Have: interpolant $\hat{f}(x) = \alpha_1 \varphi_1(x) + \cdots + \alpha_n \varphi_n(x)$

Want: integral $\int_a^b \hat{f}(x)\,dx = \int_a^b \alpha_1 \varphi_1(x) + \cdots + \alpha_n \varphi_n(x)\,dx$

$$= \alpha_1 \int_a^b \varphi_1(x)\,dx + \cdots + \alpha_n \int_a^b \varphi_n(x)\,dx$$

Example: Compute integral of quadratic interpolant on interval [0,1]



$f(x_0) = 2 \qquad f(x_1) = 0 \qquad f(x_2) = 3$

1. Find coefficients $\vec{\alpha} = V^{-1} \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$

2. Compute integrals
$$\int_0^1 1 \, dx = 1$$
$$\int_0^1 x \, dx = \frac{1}{2}$$
$$\int_0^1 x^2 \, dx = \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3}$$

3 Combine it all together

$$\underbrace{\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \end{pmatrix} V^{-1}}_{\text{"weights"}} \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$$

Demo: Compute the weights

$$\int_0^1 \hat{f}(x)\,dx = \begin{pmatrix} .166 \\ .666 \\ .166 \end{pmatrix} \cdot \begin{pmatrix} f(0) \\ f(\frac{1}{2}) \\ f(1) \end{pmatrix}$$

It turns out that this has someone's name attached to it. It's called "Simpson's Rule".

What does Simpson's Rule look like on [0, 1/2]?

$$\frac{1}{2} \begin{pmatrix} .166 \\ .666 \\ .166 \end{pmatrix} \cdot \begin{pmatrix} f(0) \\ f(\frac{1}{4}) \\ f(\frac{1}{2}) \end{pmatrix}$$

What does Simpson's Rule look like on [5, 6]?

$$\begin{pmatrix} .166 \\ .666 \\ .166 \end{pmatrix} \cdot \begin{pmatrix} f(5) \\ f(5.5) \\ f(6) \end{pmatrix}$$