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Norms

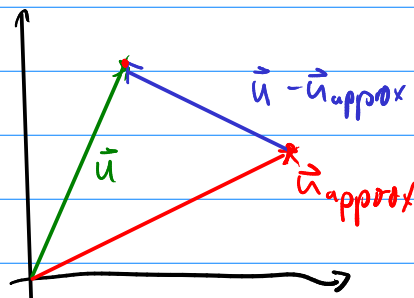
Problem with Scientific Computing: The answer always is always wrong.

Natural question: How wrong?

Need a measure of distance.

Actually: A measure of the length of a vector is fine, too. Why?

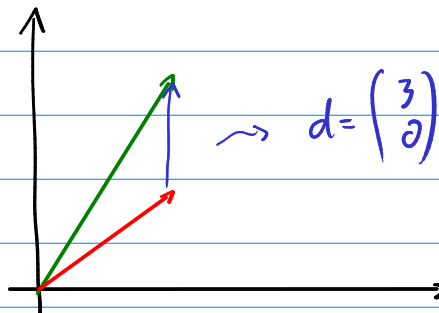
$\| \vec{x} \|$: spits out a number



$\| \vec{u} - \vec{u}_{approx} \|$ = norm of the error ✓

small, good

big, bad.



So how do we compute a 'norm'?

$$\begin{pmatrix} -3 \\ 1 \end{pmatrix} \rightarrow \|\cdot\| =$$

distance = $\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$

$|d_1| + |d_2|$ (1-norm) $\sqrt{d_1^2 + d_2^2}$ (2-norm) $\max(|d_1|, |d_2|)$ (∞ -norm)

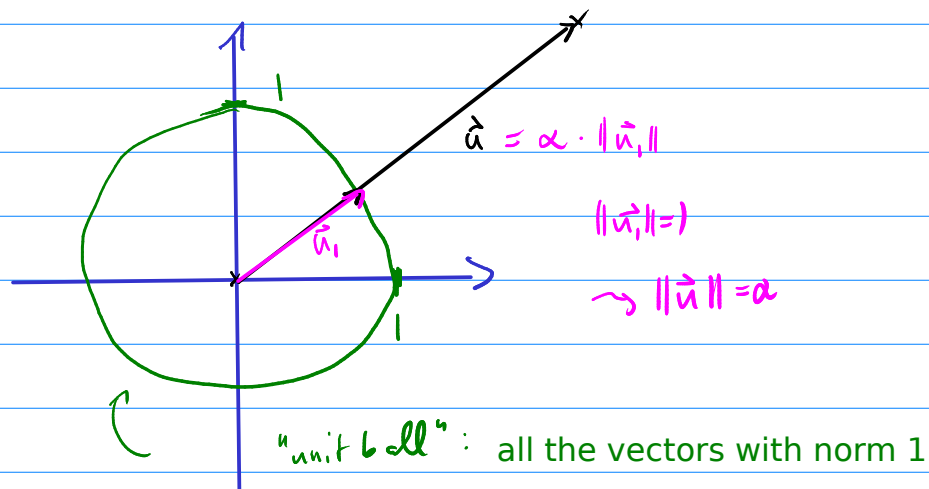
$$\vec{d} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

$$3\vec{d} = \begin{pmatrix} 3d_1 \\ 3d_2 \end{pmatrix}$$

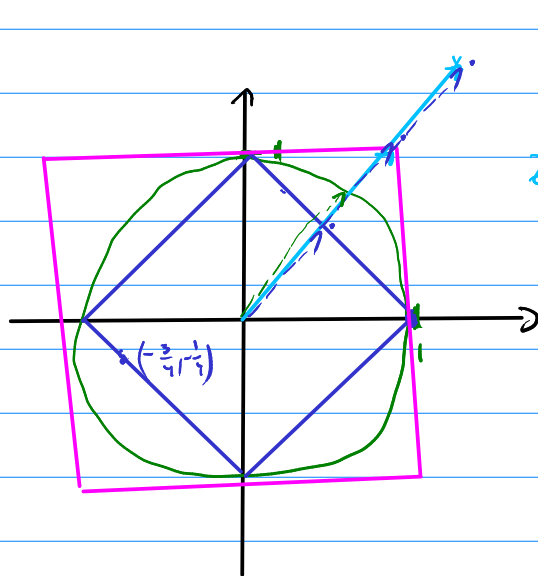
$$\|\vec{d}\| = 15$$

$$\rightarrow \|3\vec{d}\| = 3 \cdot \|\vec{d}\|$$

Get to the point.



Is the circle the only possible unit ball (in two dimensions)?



$$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$z = \begin{pmatrix} 1.5 \\ 1.5 \end{pmatrix}$$

$$\|\vec{x}\|_2 = \sqrt{x^2 + y^2} = 1$$

$$\|x\|_1 = |x| + |y| = 1$$

$$\|x\|_\infty = \max(|x|, |y|) = 1$$

$$\|z\|_\infty = 1.5$$

$$\|z\|_1 = 3$$

$$\|z\|_2 = 2.12$$

So, what does a norm have to satisfy to call itself a norm?

$$\|\vec{x}\| \geq 0$$

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\| \quad (\text{triangle inequality})$$

$$\|\alpha \vec{x}\| = |\alpha| \|\vec{x}\|$$

$$\|\vec{x}\| = 0 \Leftrightarrow \vec{x} = 0$$

Examples! I need examples!

Assume $p \geq 1$.

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\|x\|_p = \sqrt[p]{|x_1|^p + |x_2|^p + \dots + |x_n|^p}$$

$$\|x\|_1 = |x_1| + |x_2| + \dots + |x_n|$$

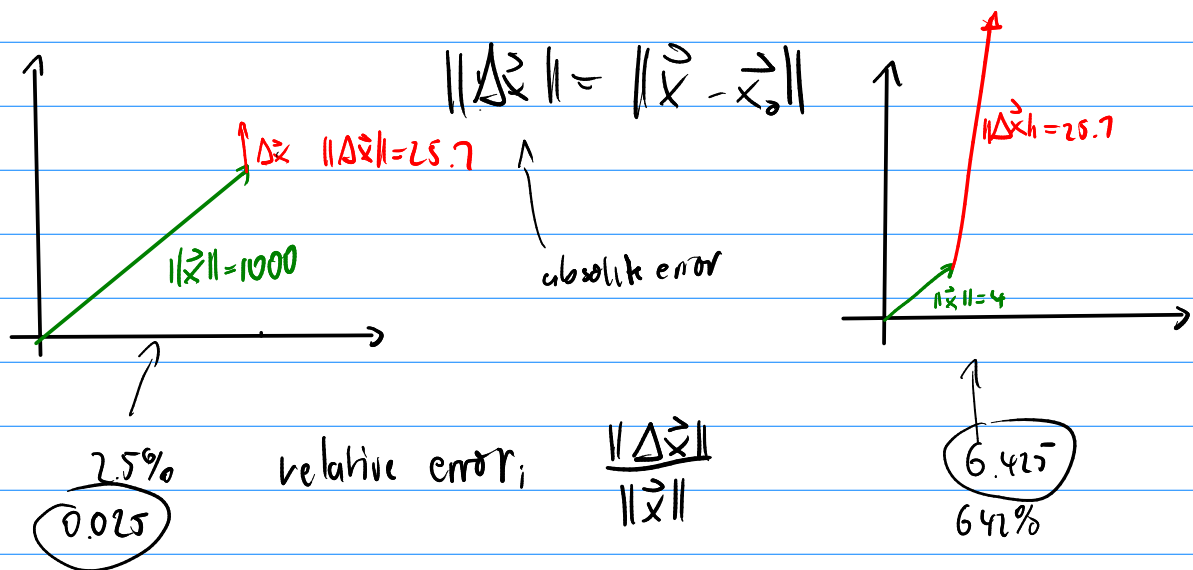
$$\|x\|_\infty =$$

Now we've put together this machinery of 'norms'. How does that help us?

thing we computed = True Value + error

$$\vec{x} = \vec{x}_0 + (\Delta \vec{x})$$

$$\Delta \vec{x} = \vec{x} - \vec{x}_0$$



another terminology: "number of accurate digits"

true:	3.14159	3141.59
computed:	3.1412	3141.2
	<u>4 accurate digits</u>	