matrix norms, elim due Feb 10.

$$
\begin{array}{lcc}
 & \vec{d}_1 & \vec{d}_2 & \vec{d}_3 \\
 & \text{Feb 4} & \text{Feb 5} & \text{Dec 27} \\
\text{Healthy} & \multirow{2}{*}{} 90\% & 70\% & 5\% \\
\text{Happy} & 100\% & 105\% & 5\%
\end{array}
$$

$$\vec{d}_1 - \vec{d}_2 = \begin{pmatrix} 20 \\ 5 \end{pmatrix} \qquad \|\vec{d}_1 - \vec{d}_2\|_\infty = 20$$

$$\vec{d}_2 - \vec{d}_3 = \begin{pmatrix} 65 \\ 100 \end{pmatrix} \qquad \| d_2 - d_3 \|_\infty = 100$$

$$\vec{u} = \overset{\text{true val.}}{\vec{u}_0} + \underbrace{\vec{\Delta u}}_{\text{abs. error}}$$

$$\text{rel. error} = \frac{\|\Delta \vec{u}\|}{\|\vec{u}_0\|}$$

The main thing we'll be studying is methods.

Methods have an input and an output. Both are inaccurate (-> have an error).

We want to say something about error at output vs error at input.

What could we say?

Model:

(Rel. error at output) = (Factor) · (Rel. error at input)

$$\text{Relative condition number} = \frac{\text{(Relative error at output)}}{\text{(Relative error in input)}}$$

$\vec{x}$: input

$\vec{y}$: output

$$\kappa = \frac{\|\Delta \vec{y}\| / \|\vec{y}\|}{\|\Delta \vec{x}\| / \|\vec{x}\|}$$

Is a condition number typically < 1?

Not typically.

Let's put condition number in terms of "digits".

If I have 3 accurate digit in my input, then my rel. error on input is 0.001.

Say my condition number is 10.

Then the relative error on the output is 0.01.

My output has 2.

Can we take norms of matrices, too?

Why does that fall short?

Suppose we have a vector with $\|x\| = 1$.

Then if we know that a matrix has norm $\|A\| = 300$.

Then we would like to be able to say that

$$\|Ax\| \leq 300 = \underbrace{\|A\|}_{300} \cdot \underbrace{\|x\|}_{1}$$

Suppose for that same matrix, I use a vector $\|y\| = 2$.

Δ ineq.

$\|x+y\| \leq \|x\| + \|y\|$

Then $\|Ay\| \leq \underbrace{\|A\|}_{300} \cdot \underbrace{\|y\|}_{2} = 600$

matrix-vec.

In the next chapter, we'll start thinking about solving systems of linear equations on a computer. What can norms and condition numbers help us say about that?

unknown = output

$$\underset{\text{output}}{A x} = \underset{\text{input}}{b} \qquad \begin{pmatrix} 2 & 5 & 1 \\ 4 & 3 & 2 \\ 1 & 1 & 1 \end{pmatrix} \cdot \vec{x} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$$Ax = b \leftarrow A(x + \Delta x) = (b + \Delta b) \qquad \text{OUT} = \kappa \cdot \text{IN}$$

$$A\Delta x = \Delta b \quad | A^{-1}\cdot$$

bonus assumption:
A is invertible

$$\cancel{A^{-1}A}\,\Delta x = A^{-1}\Delta b$$

$$\kappa = \frac{\text{rel. error in outp.}}{\text{rel. error. in inp.}} = \frac{\|\Delta x\| / \|x\|}{\|\Delta b\| / \|b\|}$$

$$= \frac{\|b\|}{\|x\|} \cdot \frac{\|\Delta x\|}{\|\Delta b\|}$$

$$= \frac{\|Ax\|}{\|x\|} \cdot \frac{\|\Delta x\|}{\|\Delta b\|}$$

$$\leq \frac{\|A\|\,\cancel{\|x\|}}{\cancel{\|x\|}} \cdot \frac{\|A^{-1}\Delta b\|}{\|\Delta b\|}$$

$$\leq \|A\| \cdot \frac{\|A^{-1}\|\,\cancel{\|\Delta b\|}}{\cancel{\|\Delta b\|}} = \|A\| \cdot \|A^{-1}\|$$

only shows "$\leq$"
but it's actually "$=$"

Can you give an example?

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \qquad \|A\|_2 = 3$$

achieved for $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$A^{-1} = \begin{pmatrix} 1/3 & 0 \\ 0 & 1 \end{pmatrix} \qquad \|A^{-1}\|_2 = 1$$

$$A\vec{x} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 3\alpha \\ \beta \end{pmatrix}$$

$$\kappa_2(A) = \|A\|_2 \cdot \|A^{-1}\|_2 \leq 3$$

icated?

What's the condition number of a 'matvec', a matrix-vector multiplication?

Solving linear systems: $\underset{\uparrow\ \text{out}}{A x} = \underset{\uparrow\ \text{in}}{b}$ $\rightsquigarrow$ $\kappa = \|A\| \cdot \|A^{-1}\|$

Matvec: $A \underset{\uparrow\ \text{in}}{x} = \underset{\uparrow\ \text{out}}{b}$ Assume A invertible $\rightsquigarrow$ $B = A^{-1}$

$| A^{-1} \cdot$

$\cancel{A^{-1}} A x = A^{-1} b$

$\underset{\uparrow\ \text{output}}{B \cdot b} = \underset{\uparrow\ \text{input}}{x}$ $\longleftarrow$ $\kappa = \|B\| \cdot \|B^{-1}\|$

$= \|A^{-1}\| \cdot \|(A^{-1})^{-1}\|$

$= \|A^{-1}\| \cdot \|A\|$

So what have we just learned?

$\|Ax\| \leq \|A\| \cdot \|x\|$

$\frac{\|Ax\|}{\|x\|} \leq \|A\|$