With enough elimination matrices, we should be able to arrive at REF...

only rewrites A

$A\vec{x} = \vec{b}$ $\qquad n^3 \to$ (1) Compute LU factorization

find $\vec{x}$ $\qquad n^2 \to$ (2) Use fw/backward subsitution

$$\underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}}_{A} \cdot \left(-\tfrac{3}{1}\right) \quad \rightsquigarrow \quad \underbrace{\begin{pmatrix} 1 & 0 \\ -\frac{3}{1} & 1 \end{pmatrix}}_{M_1}$$

$$M_3 M_2 M_1 A = U$$

$$A = M_1^{-1} M_2^{-1} M_3^{-1} U$$

What happens if we combine many elimination matrices like that?

We could rearrange that relationship to get a factorization of A!

Does an LU factorization help us solve Ax = b?

$$A\vec{x}=\vec{b} \qquad A=LU$$

$$\underbrace{LU\vec{x}}_{\vec{y}}=b \quad \rightsquigarrow \quad L\vec{y}=\vec{b}$$

fw subst

$$U\vec{x}=y$$

bw subst.

So is LU/Gaussian elimination bulletproof?

$$\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad !$$

So is our process just too stupid to find the LU factorization?

$$\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix} \cdot \begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix}$$

$$\begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$$

$\rightsquigarrow 1 \cdot u_{11} = 0 \rightsquigarrow u_{11} = 0$

$\rightsquigarrow \underbrace{\overset{0}{u_{11}} \cdot l_{21} + 0 \cdot 1 = 2}_{0 = 2}$

How are we going to fix this mess?

$$\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$$

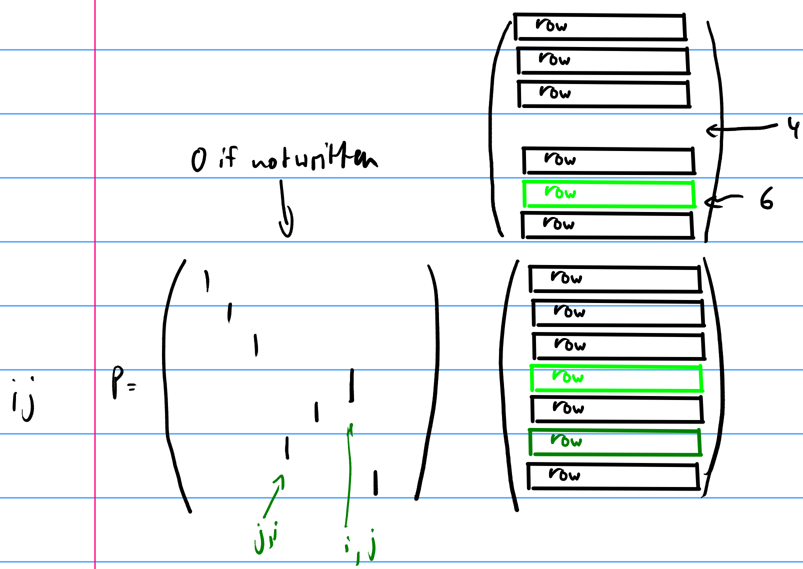
Find the biggest entry (by absolute value) in the column

we're working on ... and put that on the diagonal.

This idea is called partial pivoting or row pivoting.

The element brought onto the diagonal is called the pivot.

How do we swap rows in matrix notation?



What's the inverse of a permutation matrix (that swaps two rows)?

$$P^{-1} = P^{T}$$

How do we combine partial pivoting with the elimination matrices?

1. Pivot first col $P_1 A$
2. Eliminate first col $M_1 P_1 A$
3. Pivot second col $P_2 M_1 P_1 A$
4. Eliminate second col $M_2 P_2 M_1 P_1 A$
5. Pivot third col $P_3 M_2 P_2 M_1 P_1 A$
6. Eliminate third col $M_3 P_3 M_2 P_2 M_1 P_1 A$

That has made quite a mess of our LU factorization, right?

$$M_3 P_3 M_2 P_2 M_1 P_1 A$$

$$\cancel{P_3 P_2 P_1 \; M_3 M_2 M_1 = L\,?}$$

So... how do we sort out this mess?

So... how do we sort out this mess? (cont'd)

$$PA = LU$$

$$M_3 P_3 M_2 P_2 M_1 P_1 A$$

Define: $L_3 = M_3$

$$L_2 = P_3 M_2 P_3^{-1}$$

$$L_1 = P_3 P_2 M_1 P_2^{-1} P_3^{-1}$$

Then: $L_3 L_2 L_1 P_3 P_2 P_1$

$$= M_3 P_3 M_2 \cancel{P_3^{-1}} \cancel{P_3} P_2 M_1 \cancel{P_2^{-1}} \cancel{P_3^{-1}} \cancel{P_3} \cancel{P_2} P_1$$

$$= M_3 P_3 M_2 P_2 M_1 = \underbrace{M_3 P_3 M_2 P_2 M_1 P_1 A}_{C}$$

$$C \cdot A = U$$

$$L_3 L_2 L_1 P_3 P_2 P_1 A = U$$

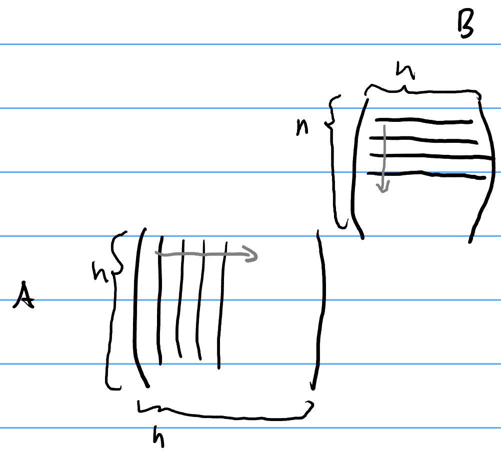
$$\underbrace{P_3 P_2 P_1}_{P} A = \underbrace{L_1^{-1} L_2^{-1} L_3^{-1}}_{L} U$$

$$PA = LU$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} \to 0 \quad a \quad = 7$$

Let's talk about computational cost. What is the asymptotic cost of multiplying two n x n matrices?



So how expensive is LU factorization?

Can LU deal with non-square matrices?

4 Applications of LU

(1) Solve linear equations. How?

(2) Solve a matrix equation. How?

(3) Find the basis of a span. How?

(4) Find the determinant of a matrix. How?

(5) We'd like to find the rank* of a matrix. Is that possible using a computer?

*rank: Number of linearly independent rows/columns

Suppose we take that into account. How would we compute the rank?

(6) Finding the nullspace of a matrix A

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