Numerical Methods (CS 357)

Worksheet

Problem 1. Permutation Matrices

Create a permutation `P` matrix that takes the vector $x = [0, 1, 2, 3, 4]^T$ to $Px = [1, 3, 4, 0, 2]$.

```
import numpy as np
P = np.zeros((5,5))

P[   ,0] = 1
P[   ,1] = 1
P[   ,2] = 1
P[   ,3] = 1
P[   ,4] = 1

print(P.dot(x))
```

Problem 2. Pivoted LU

Factor the matrix

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 3 \\ 2 & 4 & 4 \end{bmatrix}$$

into a permutation matrix P, a lower triangular matrix L, and an upper triangular matrix U. Here are a few reminders about the process (so that you don't have to go look these up):

- Original factorization: $M_2P_2M_1P_1A = A$ U
- $L_2 = M_2$
- $L_1 = P_2M_1P_2^{-1}$
- $L = L_1^{-1}L_2^{-1}$
- $P = P_2P_1$

```
import numpy as np

P = np.zeros((3,3), dtype=np.float64)
P[   , 0] = 1
P[   , 1] = 1
P[   , 2] = 1

L = np.array([
    [1, 0, 0],
    [ , 1, 0],
    [ ,  , 1],
    ])

U = np.array([
    [ ,  ,  ],
    [0,  ,  ],
    [0, 0,  ],
    ])

print(P.dot(A)-L.dot(U))
```

$$A = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 1 & 3 \\ 2 & 4 & 4 \end{pmatrix}$$

$$P_1 = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} \qquad P_1A = \begin{pmatrix} 2 & 4 & 4 \\ 1 & 1 & 3 \\ 0 & 2 & 1 \end{pmatrix} \cdot -\tfrac{1}{2}$$

$$M_1 = \begin{pmatrix} 1 & & \\ -\frac{1}{2} & 1 & \\ 0 & 0 & 1 \end{pmatrix} \qquad M_1P_1A = \begin{pmatrix} 2 & 4 & 4 \\ 0 & -1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad P_2M_1P_1A = \begin{pmatrix} 2 & 4 & 4 \\ 0 & 2 & 1 \\ 0 & -1 & 1 \end{pmatrix} \cdot \tfrac{1}{2}$$

$$M_2 = \begin{pmatrix} 1 & & \\ & 1 & \\ & \frac{1}{2} & 1 \end{pmatrix} \qquad U = M_2P_2M_1P_1A = \begin{pmatrix} 2 & 4 & 4 \\ 0 & 2 & 1 \\ 0 & 0 & 1.5 \end{pmatrix}$$

$$L_2 = M_2 = \begin{pmatrix} 1 & & \\ & 1 & \\ & \frac{1}{2} & 1 \end{pmatrix} \qquad L_1 = P_2 \begin{pmatrix} 1 & & \\ -\frac{1}{2} & 1 & \\ 0 & 0 & 1 \end{pmatrix} P_2^{-1} = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ -\frac{1}{2} & 0 & 1 \end{pmatrix}$$

$$L = L_1^{-1}L_2^{-1} = \begin{pmatrix} 1 & & \\ & 1 & \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}$$

$$P = P_2P_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$